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Robust Control for Variable Order Time Fractional Butterfly-Shaped Chaotic Attractor System

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ABSTRACT

In this article, we investigated the robust control approach for variable-order fractional time fractional butterfly- shaped chaotic attractor system that the fractional derivative is considered in Atangana-Baleanu-Caputo sense. We show the computational algorithm with high accuracy for solving the proposed systems. For the suggested system, Adams-Bashforth-Moulton approach applied for converting the system of the equations into a system of linear or nonlinear algebraic equations. The existence and uniqueness of the solution are shown and also asymptotically stable is investigated in this article. At the end, a number of statistical indicators were presented in order efficiency, accuracy and simple applicability of the proposed method.

1. Introduction

By means of Fractional Calculus (FC), the integration operators and differentiation operators achieve fractional order. In recent decades, the study of FC has absorbed growing attention as hot research topic on a global scale [1-14]. Samko extended the constant order FC in an outstanding manner [15].

In this research work, fractional operators in which order is considered to be a function of time, space or a few other variables are proposed. Noticeable applications of such fractional variable-order operators are introduced in [16-18]. Due to the fact that finding the exact solutions of variable-order fractional differential equations is impossible, devising numerical schemes in order to solve these equations is an important research topic. Adams-Bashforth's method is known to, from a conventional

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point of view, as powerful and excellent numerical method that can present a numerical solution of fractional differential equations [19-22].

Recently the authors have developed a constant-order numerical scheme- in [23]– that is able to combine the fundamental theorem of fractional calculus and the two-step Lagrange polynomial. Drawing on this method, the present paper generalizes the numerical schemes that were introduced in [23] to simulate variable-order fractional differential operators with power-law, exponential-law and Mittag-Leffler kernel. Stability analysis, dynamical properties and simulation of some fractional differential equations are dealt with in [24-27].

Nonlinear systems, form a holistic point of view, are not in line with the superposition principle. Mathematically speaking, a nonlinear system is a problem, in which the variables that are supposed to be solved are not writable as an independent component linear combination. In case the equation involves a nonlinear function (power or cross product), the system is also considered nonlinear. Moreover, if the system is characterized with a nonlinear transfer, for instance a diode current-voltage characteristic, it is then regarded as nonlinear. Most importantly, we must refer to typical nonlinearity. Moreover, the system is nonlinear provided that a typical amount of nonlinearity like saturation, hysteresis, etc exists. Characteristics like this are the fundamental attributes of a nonlinear system. As the majority of real physical systems are, in essence, nonlinear, nonlinear systems have fascinated engineers, physicists and mathematicians. Solving non-linear equations by means of analytical methods would prove difficult and arise remarkable phenomena such as chaos and bifurcation. Even simple nonlinear (or piecewise linear) dynamical systems may behave entirely unpredictably, referred to as deterministic chaos. Owing to the fact that trivial systems can also involve chaos, chaos theory has become highly important. It is noteworthy, at this point, no unique definition is given for chaos. Chaotic dynamics are most commonly the ones originating from regular dynamical equations that do not include stochastic coefficients, yet simultaneously have trajectories the same as or indistinguishable from a number of stochastic processes. Some definitions are given for chaotic dynamics, e.g., (i) system characterized with minimum one positive Lyapunov exponent is called chaotic, (ii) a system characterized with positive entropy is called chaotic, and (iii) a system that is equivalent to hyperbolic or Anosov system is called chaotic, and so on. What all these definitions have in common is that local instability and divergence of initially close trajectories exist there. Nonetheless, the definitions are not entirely similar in their sense.

This paper mainly handles robust control. This issue is then briefly discussed here. Sectors like car production industry, mining, and other hardware make widespread use of feedback control systems. To meet the ever-growing demands for higher reliability and better efficiency levels, such control systems are constantly obliged to accomplish more accurate and desirable general performance in response to the ever-changing and challenging operating circumstances. For designing control systems to achieve more desirable robustness and efficiency while monitoring complex procedures, control engineers need novel designing apparatus and a more effective control theory. A standard technique of enhancing a control system performance is adding more sensors and actuators. As a result, a Multi-Input Multi-Output (MIMO) control system will necessarily be obtained. Therefore, each methodology related to designing modern feedback control systems must be capable of managing the issue of multiple actuators and sensors. A control system is also robust when: (1) its sensitivity level is low, (2) over a range of parameter variations, it remains stable, and (3) the performance invariably meets the specifications in the presence of a set of system parameter variations [28-32].

The present paper is outlined as follows: Some required preliminaries in the sequel are presented in Section 2. Section 3 deals with the existence and uniqueness of solutions. Section 4 deals with the numerical approach procedure. Robust control for variable order time fractional butterfly- shaped chaotic attractor system is discussed in Section 5. Section 6 presents simulation results. Finally, the method and the generated results are briefly discussed in Section 7.

2. Preliminaries

Some basic tools that will be needed in future are given in this section. The Atangana-Baleanu fractional derivative with variable- $\alpha(t)$ order in Liouville-Caputo sense (ABC) is defined [33] as

$$ {}^{ABC}_{0}D_t^{\alpha(t)} f(t) = \frac{B(\alpha(t))}{1-\alpha(t)} \int_0^t E_{\alpha(t)}\left[-\alpha(t)\frac{(t-\tau)^{\alpha(t)}}{1-\alpha(t)}\right] f'(\tau)d\tau, n-1<\alpha(t)\le n, \quad (1)$$

Where $B(\alpha(t)) = 1-\alpha(t)+\frac{\alpha(t)}{\Gamma(\alpha(t))}$ is a normalization function. Related integral Atangana-Baleanu can be formulated as

$$ {}^{ABC}_{0}I_t^{\alpha(t)} f(t) = \frac{1-\alpha(t)}{B(\alpha(t))} f(t) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t f(\tau)(t-\tau)^{\alpha(t)-1}d\tau, \; n-1<\alpha(t)\le n. \quad (2)$$

Consider n^{th} an -order fractional differential equation of the form

$$ {}^{ABC}_{0}D_t^{\alpha(t)} f(t) = G(t, f(t)), \quad f^{(k)}(0) = f_0^k, k=0,1,\dots,n-1. \quad (3)$$

This equation can be rewritten as

$$ f(t) = f_0 + \frac{1-\alpha(t)}{B(\alpha(t))} G(t, f(t)) + \frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t G(\tau, f(\tau))(t-\tau)^{\alpha(t)-1}d\tau. \quad (4)$$

The Adams method can be extended for this equation as follows:

$$ f_{i+1}^{P} = f_0 + \frac{1-\alpha(t_i)}{B(\alpha(t_i))} G(t_i, f_i) + \frac{\alpha(t_i)}{B(\alpha(t_i))\Gamma(\alpha(t_i))}\sum_{j=0}^{i} b_{j.i+1} G(t_i + f_i), \quad (5)$$

$$ f_{i+1} = f_0 + \frac{1-\alpha(t_{i+1})}{B(\alpha(t_{i+1}))} G(t_i+1, f_{i+1}^{p}) + \frac{\alpha(t_{i+1}) h^{\alpha(t_i+1)}}{B(\alpha(t_{i+1}))\Gamma(\alpha(t_{i+1})+2)} \times $$

$$\left[G\left(t_{i+1}, f_{i+1}^{p}\right)+\sum_{j=0}^{i} a_{j,i+1} G\left(t_j+f_j\right)\right],$$

where

$$b_{j,i+1}=\frac{h^{\alpha(t_{i+1})}}{\alpha(t_{i+1})}\left((i-j+1)^{\alpha(t_{i+1})}-(i-j)^{\alpha(t_{i+1})}\right), \quad j=0,1,..i, \tag{6}$$

and

$$a_{j,i+1}\begin{cases} i^{\alpha(t_{i+1})+1}-\left(i-\alpha\left(t_i+1\right)\right)(i+1)^{\alpha(t_{i+1})}, & j=0. \\ (i-j+2)^{\alpha(t_{i+1})+1}+(i-j)^{\alpha(t_{i+1})+1}-2(i-j+1)^{\alpha(t_{i+1})+1}, & 1\leq j\leq i. \end{cases} \tag{7}$$

3. Existence and Uniqueness of Solutions under Atangana Baleanu Fractional Derivative with Variable-Order $\alpha(t)$

In this section, we use well-known fixed point technique for the existence of solutions of ***Eq. (3)***. Thus, by taking AB-fractional integral operator of variable-order $\alpha(t)$ which is given in ***Eq. (2)***, we obtain

$$f(t)-f(0)=\frac{1-\alpha(t)}{B(\alpha(t))}G(t,f(t))+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}G(\theta,f(\theta))\,d\theta. \tag{8}$$

We assume that, for continuous functions G and $f_1 \in L[0,1]$, there exist a constant λ_1 such that the following hold true

$$\|G(t,f)-G(t,f_1)\|\leq\lambda_1\|f-f_1\|. \tag{9}$$

From ***Eq. (8)***, we define the following recursive relation

$$f_n(t)-f(0)=\frac{1-\alpha(t)}{B(\alpha(t))}G(t,f_{n-1}(t))+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}G(\theta,f_{n-1}(\theta))\,d\theta. \tag{10}$$

With initial condition $f_0(t)=f(0)$, we consider the following differences

$$\begin{aligned}\psi^{n+1}(t)=(f_{n+1}-f_n)(t)=&\frac{1-\alpha(t)}{B(\alpha(t))}(G(t,f_n(t))-G(t,f_{n-1}(t))\\&+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f_n(\theta))-G(\theta,f_{n-1}(\theta))\,d\theta.\end{aligned} \tag{10}$$

Take the norm of ***Eq. (11)***, then we have

$$\begin{aligned}\left\|\psi^{n+1}(t)\right\| = \left\|(f_{n+1}-f_n)(t)\right\| = \Bigg\| & \frac{1-\alpha(t)}{B(\alpha(t))}(G(t,f_n(t))-G(t,f_{n-1}(t)) \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f_n(\theta))-G(\theta,f_{n-1}(\theta))d\theta\Bigg\| \\ & \leq \frac{1-\alpha(t)}{B(\alpha(t))}\left\|(G(t,f_n(t))-G(t,f_{n-1}(t))\right\| \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\left\|\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f_n(\theta))-G(\theta,f_{n-1}(\theta))d\theta\right\| \\ & \leq \frac{1-\alpha(t)}{B(\alpha(t))}\left\|(G(t,f_n(t))-G(t,f_{n-1}(t))\right\| \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}\left\|G(\theta,f_n(\theta))-G(\theta,f_{n-1}(\theta)\right\|d\theta.\end{aligned} \quad (11)$$

Applying ***Eq. (12)*** and we prove the existence of solution for ***Eq. (3)***. For this aim, we define the function $\phi_n(t)=(f_{n+1}-f)(t)+f(0)$. Then, using ***Eq. (12)***, we obtain

$$\begin{aligned}\left\|\phi_n(t)\right\| = \Bigg\| & \frac{1-\alpha(t)}{B(\alpha(t))}(G(t,f_n(t))-G(t,f(t)) \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f_n(\theta))-G(\theta,f(\theta))d\theta\Bigg\| \\ & \leq \frac{1-\alpha(t)}{B(\alpha(t))}\left\|(G(t,f_n(t))-G(t,f(t))\right\| \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\left\|\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f_n(\theta))-G(\theta,f(\theta))d\theta\right\| \\ & \leq \frac{1-\alpha(t)}{B(\alpha(t))}\left\|(G(t,f_n(t))-G(t,f(t))\right\| \\ & +\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}\left\|G(\theta,f_n(\theta))-G(\theta,f(\theta)\right\|d\theta \\ & \leq \frac{1-\alpha(t)}{B(\alpha(t))}\lambda_1\left\|f_n(t)-f(t)\right\|+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}\lambda_1\left\|f_n(t)-f(t)\right\| \\ & \leq \left[\frac{1-\alpha(t)}{B(\alpha(t))}+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}\right]^n\left\|f(t)-f_1(t)\right\|\lambda_1^{\,n}.\end{aligned} \quad (13)$$

Then ***Eq. (13)***, shows that the function $lim_{n\to\infty}\phi_n(t)=0$ for $\frac{1-\alpha(t)}{B(\alpha(t))}+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}<1$ and $\lambda_1<1$ which further shows that $lim_{n\to\infty} f_{n+1}(t)=f(t)$. Thus, solutions of the ***Eq. (3)*** exist.

For analysis of the uniqueness of solutions of the model ***Eq. (3)***, we consider the contrary path for the proof. That is, let there exist another pair of solutions (f_1,f_2) of ***Eq. (3)*** satisfying the integral system given as

$$f_1(t)-f_1(0)=\frac{1-\alpha(t)}{B(\alpha(t))}G(t,f_1(t))+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}G(\theta,f_1(\theta))\,d\theta. \quad (14)$$

For the model ***Eq. (3)***, we consider $f_1(0)=0$, then we have

$$\begin{aligned}\|(f-f_1)(t)\| &= \Big\|\frac{1-\alpha(t)}{B(\alpha(t))}(G(t,f(t))-G(t,f_1(t))\\ &\quad+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f(\theta))-G(\theta,f_1(\theta))\,d\theta\Big\|\\ &\le \frac{1-\alpha(t)}{B(\alpha(t))}\|(G(t,f(t))-G(t,f_1(t))\|\\ &\quad+\frac{\alpha(t)}{B(\alpha(t))\Gamma(\alpha(t))}\Big\|\int_0^t (t-\theta)^{\alpha(t)-1}(G(\theta,f(\theta))-G(\theta,f_1(\theta))\,d\theta\Big\|\\ &\le \frac{1-\alpha(t)}{B(\alpha(t))}\lambda_1\|f-f_1\|\\ &\quad+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}\lambda_1\|f-f_1\|\\ &=[\frac{1-\alpha(t)}{B(\alpha(t))}+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}]\lambda_1\|f-f_1\|.\end{aligned} \quad (15)$$

Which implies

$$\|(f-f_1)(t)\|(1-[\frac{1-\alpha(t)}{B(\alpha(t))}+\frac{1}{B(\alpha(t))\Gamma(\alpha(t))}]\lambda_1)\le 0. \quad (16)$$

Using ***Eq. (16)***, we obtain $\|(f-f_1)(t)\|\to 0$. Consequently, the solution of ABC-fractional order which is given by ***Eq. (3)***, is unique.

4. Numerical Approach

This paper discusses a recently presented butterfly-shaped chaotic attractor system with six terms such as three multipliers for presenting the necessary nonlinearity for the folding trajectories [34, 35]. The numerical simulation and theoretical analysis demonstrate vividly that the new system is similar to Lorenz and other chaotic attractors. However, its topological structure differs from any chaotic attractors that exist. Adams method for the butterfly-shaped chaotic attractor system is developed here

$$\begin{cases} {}^{ABC}_{\ \ 0}D_t^{\alpha(t)}x(t)=G_1(t_i,x_i,y_i,z_i)=a(y-x)(t),\\ {}^{ABC}_{\ \ 0}D_t^{\alpha(t)}y(t)=G_2(t_i,x_i,y_i,z_i)=x(t)z(t)+by(t),\\ {}^{ABC}_{\ \ 0}D_t^{\alpha(t)}z(t)=G_3(t_i,x_i,y_i,z_i)=-x^2(t)-cz(t),\end{cases} \quad (17)$$

The same as the previous section, we have

$$\begin{cases} x_{i+1}^{p} = x_0 + \frac{1-\alpha(t_i)}{B(\alpha(t_i))} G_1(t_i, x_i, y_i, z_i) + \frac{\alpha(t_i)}{B(\alpha(t_i))\Gamma(\alpha(t_i))} \sum_{j=0}^{i} b_{j.i+1} G_1(t_j, x_j, y_j, z_j) \\ y_{i+1}^{p} = y_0 + \frac{1-\alpha(t_i)}{B(\alpha(t_i))} G_2(t_i, x_i, y_i, z_i) + \frac{\alpha(t_i)}{B(\alpha(t_i))\Gamma(\alpha(t_i))} \sum_{j=0}^{i} b_{j.i+1} G_2(t_j, x_j, y_j, z_j) \\ z_{i+1}^{p} = z_0 + \frac{1-\alpha(t_i)}{B(\alpha(t_i))} G_3(t_i, x_i, y_i, z_i) + \frac{\alpha(t_i)}{B(\alpha(t_i))\Gamma(\alpha(t_i))} \sum_{j=0}^{i} b_{j.i+1} G_3(t_j, x_j, y_j, z_j). \end{cases} \tag{18}$$

And

$$\begin{cases} x_{i+1} = x_0 + \frac{1-\alpha(t_{i+1})}{B(\alpha(t_{i+1}))} G_1(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \frac{\alpha(t_{i+1}) h^{t_i+1}}{B(\alpha(t_{i+1}))\Gamma(\alpha(t_{i+1})+2)} \times \\ \qquad \left[G_1(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \sum_{j=0}^{i} a_{j.i+1} G_1(t_j, x_j, y_j, z_j)\right], \\ y_{i+1} = y_0 + \frac{1-\alpha(t_{i+1})}{B(\alpha(t_{i+1}))} G_2(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \frac{\alpha(t_{i+1}) h^{t_i+1}}{B(\alpha(t_{i+1}))\Gamma(\alpha(t_{i+1})+2)} \times \\ \qquad \left[G_2(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \sum_{j=0}^{i} a_{j.i+1} G_2(t_j, x_j, y_j, z_j)\right], \\ z_{i+1} = z_0 + \frac{1-\alpha(t_{i+1})}{B(\alpha(t_{i+1}))} G_3(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \frac{\alpha(t_{i+1}) h^{t_i+1}}{B(\alpha(t_{i+1}))\Gamma(\alpha(t_{i+1})+2)} \times \\ \qquad \left[G_3(t_{i+1}, x_{i+1}^{p}, y_{i+1}^{p}, z_{i+1}^{p}) + \sum_{j=0}^{i} a_{j.i+1} G_3(t_j, x_j, y_j, z_j)\right], \end{cases} \tag{19}$$

that is an iterative technique to obtain a solution for this fractional problem.

5. Robust Control for Variable Order Time Fractional Butterfly- Shaped Chaotic Attractor System

If two exactly similar copies of a chaotic system start in similar initial conditions, they will not have the same motions for a long period of time. The exponential divergence of orbits will amplify all of the initial minor errors. Apparently, it will firstly prove very difficult to keep the two chaotic system copies synchronized. Since then, chaotic dynamical system synchronization has been under extensive scientific examination. Identical synchronization is, in principle, to take two copies of a fixed chaotic system and to make one of them take control of the other. The master or drive system creates a signal to feed the slave or response system subsequently.

The signal is normally one of the applied coordinates in explaining the chaotic system. Synchronization can be considered as a form of chaos control and the simplicity of the coupling mechanism would make multiple applications possible.

Let us consider the system [36-42]

$$\begin{cases} {}^{ABC}_{\ \ 0}D_t^{\alpha(t)} x(t) = a(y-x)(t), \\ {}^{ABC}_{\ \ 0}D_t^{\alpha(t)} y(t) = x(t)z(t) + by(t), \\ {}^{ABC}_{\ \ 0}D_t^{\alpha(t)} z(t) = -x^2(t) - cz(t). \end{cases} \tag{20}$$

Where the variable x, y, z denotes the states, and a, b, c are positive parameters.

$$\begin{cases} ABC_0D_t^{\alpha(t)}x(t) = a(y-x)(t) + u_x, \\ ABC_0D_t^{\alpha(t)}y(t) = x(t)z(t) + by(t) + u_y, \\ ABC_0D_t^{\alpha(t)}z(t) = -x^2(t) - cz(t) + u_z. \end{cases} \quad (21)$$

Robust control is aimed at suppressing the chaotic behavior in the systems. $\bar{x},\bar{y}$ and $\bar{z}$ are defined as the auxiliary system equilibrium points.

$$\begin{cases} ABC_0D_t^{\alpha(t)}\bar{x}(t) = a(\bar{y}-\bar{x})(t), \\ ABC_0D_t^{\alpha(t)}\bar{y}(t) = \bar{x}(t)\bar{z}(t) + b\bar{y}(t), \\ ABC_0D_t^{\alpha(t)}\bar{z}(t) = -\bar{x}^2(t) - c\bar{z}(t). \end{cases} \quad (22)$$

Now, the control errors are defined as

$$\begin{cases} e_x(t) = x(t) - \bar{x}(t), \\ e_y(t) = y(t) - \bar{y}(t), \\ e_z(t) = z(t) - \bar{z}(t), \end{cases} \quad (23)$$

then

$$\begin{cases} {}^{ABC}{}_0D_t^{\alpha(t)}e_x(t) = a(e_y - e_x) + u_x, \\ {}^{ABC}{}_0D_t^{\alpha(t)}e_y(t) = e_x(e_z + \bar{z}) + be_y + u_y, \\ {}^{ABC}{}_0D_t^{\alpha(t)}e_z(t) = -e_x(e_x + \bar{x}) - ce_z + u_z. \end{cases} \quad (24)$$

As mentioned in the previous section, the systems' chaos behavior will be suppressed. The equilibrium points are then defined as equal to zero *i.e* $\bar{x} = \bar{y} = \bar{z} = 0$. ***Eq. (24)***,

$$\begin{cases} {}^{ABC}{}_0D_t^{\alpha(t)}e_x(t) = -ae_y - ae_x + u_x, \\ {}^{ABC}{}_0D_t^{\alpha(t)}e_y(t) = e_xe_z + be_y + u_y, \\ {}^{ABC}{}_0D_t^{\alpha(t)}e_z(t) = -e_x^2 - ce_z + u_z. \end{cases} \quad (25)$$

From ***Eq. (25)***, the control law is defined as

$$\begin{cases} u_x = -ae_y + xe_x - k_xe_x, \\ u_y = -e_xe_z - be_y - k_ye_y, \\ u_z = +e_x^2 + ce_z - k_ze_z. \end{cases} \quad (26)$$

Where $k_x, k_y, k_z > 0$.

Theorem 1. On condition that the control is defined as ***Eq. (26)***, the ***system (20)*** is asymptotically stable.

Proof. The controller ***stability (26)*** will be provable through the following Lyapanov function

$$V(t)=\frac{1}{2}\left(e_x^2+e_y^2+e_z^2\right)$$

Then the derivative of the Lyapanov function is given by

$$ {}^{ABC}_{0}D_t^{\alpha(t)}V(t)=e_x\,{}^{ABC}_{0}D_t^{\alpha(t)}e_x+e_y\,{}^{ABC}_{0}D_t^{\alpha(t)}e_y+e_z\,{}^{ABC}_{0}D_t^{\alpha(t)}e_z. \quad (27)$$

From ***Eqs. (25)*** and ***(26)***, the dynamic of each control error can be defined as

$$\begin{cases} {}^{ABC}{}_{0}D_t^{\alpha(t)}e_x(t)=ae_y-ae_x-ae_y+ae_x-k_xe_x, \\ {}^{ABC}{}_{0}D_t^{\alpha(t)}e_y(t)=e_xe_z+be_y-e_xe_z-be_y-k_ye_y, \\ {}^{ABC}{}_{0}D_t^{\alpha(t)}e_z(t)=-e_x^2-ce_z+e_x^2+ce_z-k_ze_z, \end{cases} \quad (28)$$

Therefore

$$\begin{cases} {}^{ABC}_{0}D_t^{\alpha(t)}e_x(t)=-k_xe_x, \\ {}^{ABC}_{0}D_t^{\alpha(t)}e_y(t)=-k_ye_y, \\ {}^{ABC}_{0}D_t^{\alpha(t)}e_z(t)=-k_ze_z, \end{cases} \quad (29)$$

Substituting ***Eq. (29)*** into ***Eq. (28)*** we have

$$ {}^{ABC}_{0}D_t^{\alpha(t)}V(t)=-k_xe_x^2-k_ye_y^2-k_ze_z^2\le 0. \quad (30)$$

The fact that $k_x, k_y, k_z > 0$ guarantees that the derivative of the Lyapunov function will always be negative or equal to zero causing asymptomatic stability.

6. Simulation Results

This section is to generalize the numerical scheme for the fractional butterfly- shaped chaotic attractor system in Atangana-Baleanu-Caputo fractional derivatives with variable order $\alpha(t)$ i. One of the initial motivations of the fractional control presented here is to promote the performance adjustment flexibility.

After implementing the robust controller proposed with k_y, ***system (20)*** changes to

$$\begin{cases} {}^{ABC}{}_{0}D_t^{\alpha(t)}x(t)=a(y-x)(t), \\ {}^{ABC}{}_{0}D_t^{\alpha(t)}y(t)=x(t)z(t)+b_y(t)-k_yy(t), \\ {}^{ABC}{}_{0}D_t^{\alpha(t)}z(t)=-x^2(t)-cz(t). \end{cases}$$

The numerical results are here obtained assuming $x_0 = 1$, $y_0 = 2$ and $z_0 = 10$. The problem parameters are a = 30, b =15, c = 11. In Fig. 1, the phase diagram is drawn for the fixed differential order $\alpha(t)=1$ and robust controller with similar derivative order is drawn in ***Fig. 2*** $Ky=5$. The system simulation was performed over 200 seconds.

The phase diagram and robust controller with $Ky=5$ are drawn assuming $\alpha(t)=\tanh(\frac{\pi}{2}+t)$ in ***Figs. 3*** and ***4***, respectively. From the figure, it is obvious that the Adams-Bashforth-Moulton method is capable of solving the variable-order fractional differential equation simply and effectively. Furthermore, compared to ***Figs. 1*** and ***2***, it is observed that the derivative order has profound effects on the system results.

The analysis given above implies that designers could reach the suitable dynamic behavior of turbofan and expenses through determining the proper fractional control.

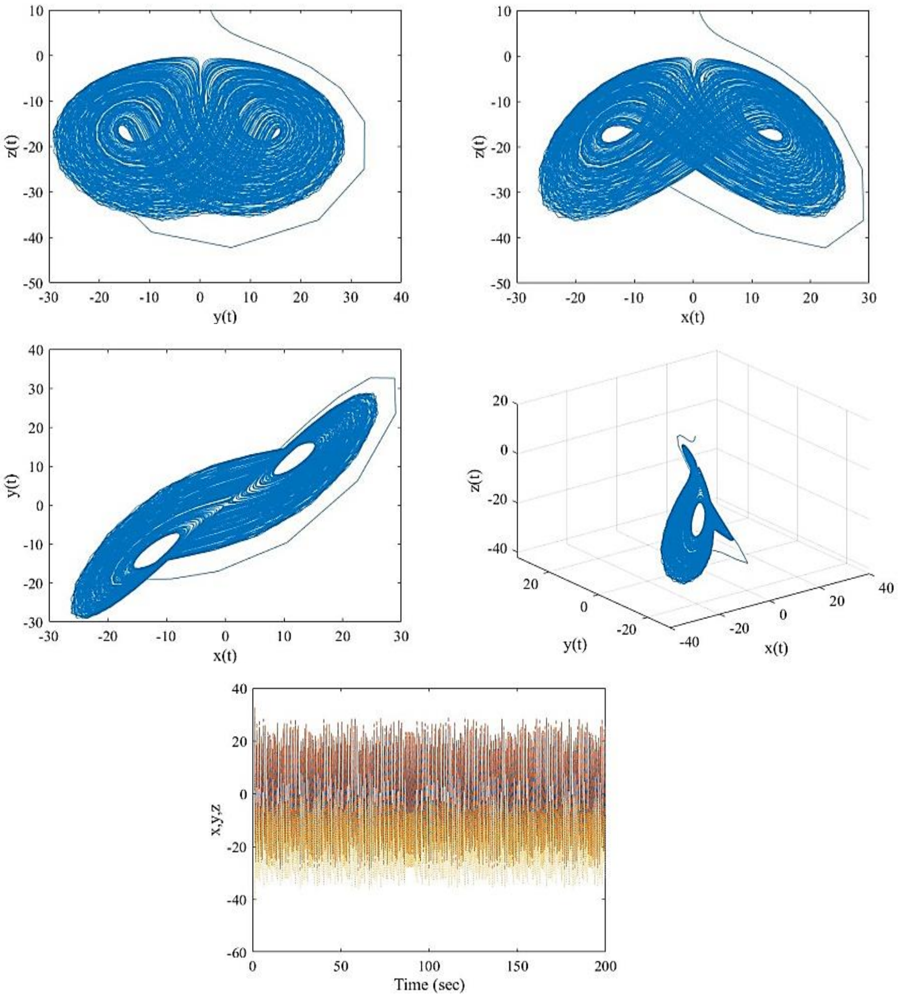


Fig. 1. *The system's phase diagram for the order* $\alpha(t)=1$.

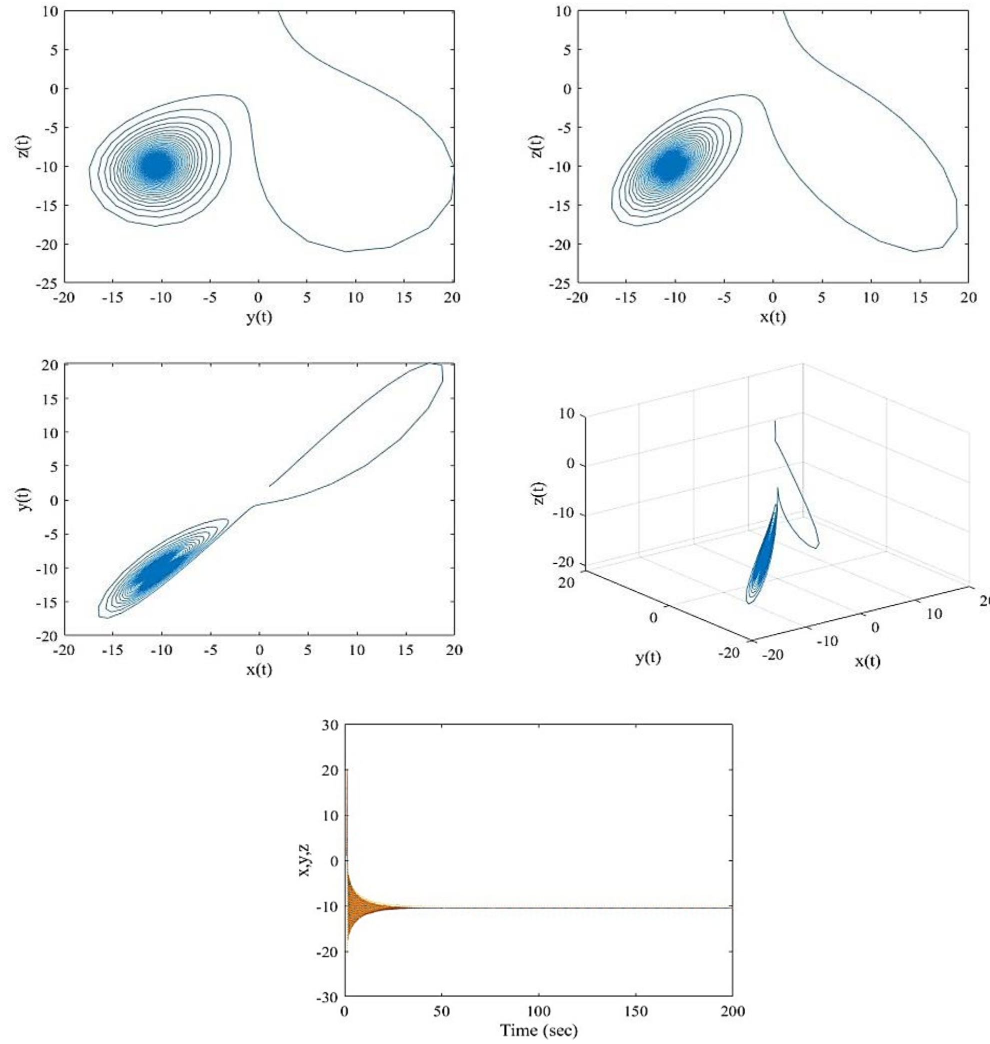
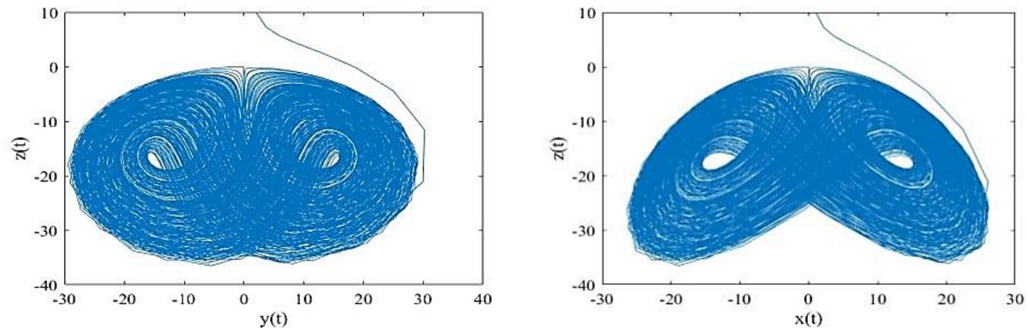


Fig. 2. *The proposed system's robust controller for the order* $\alpha(t)=1$.



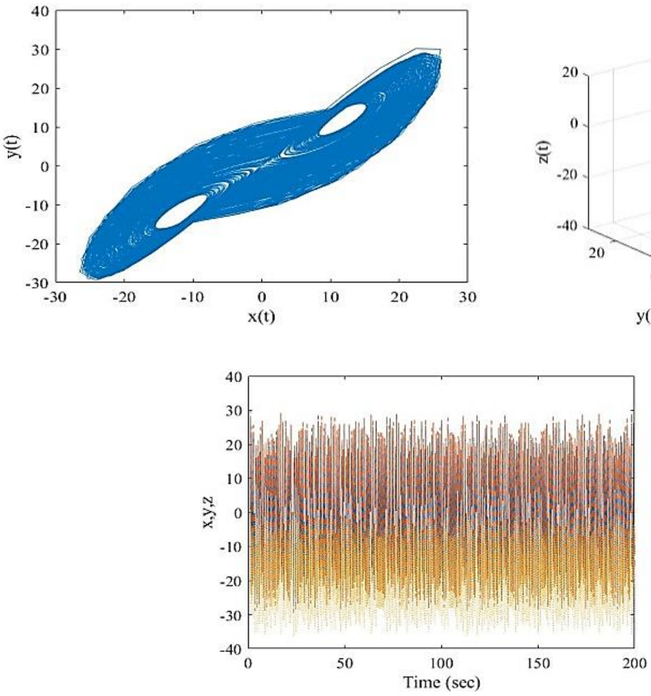
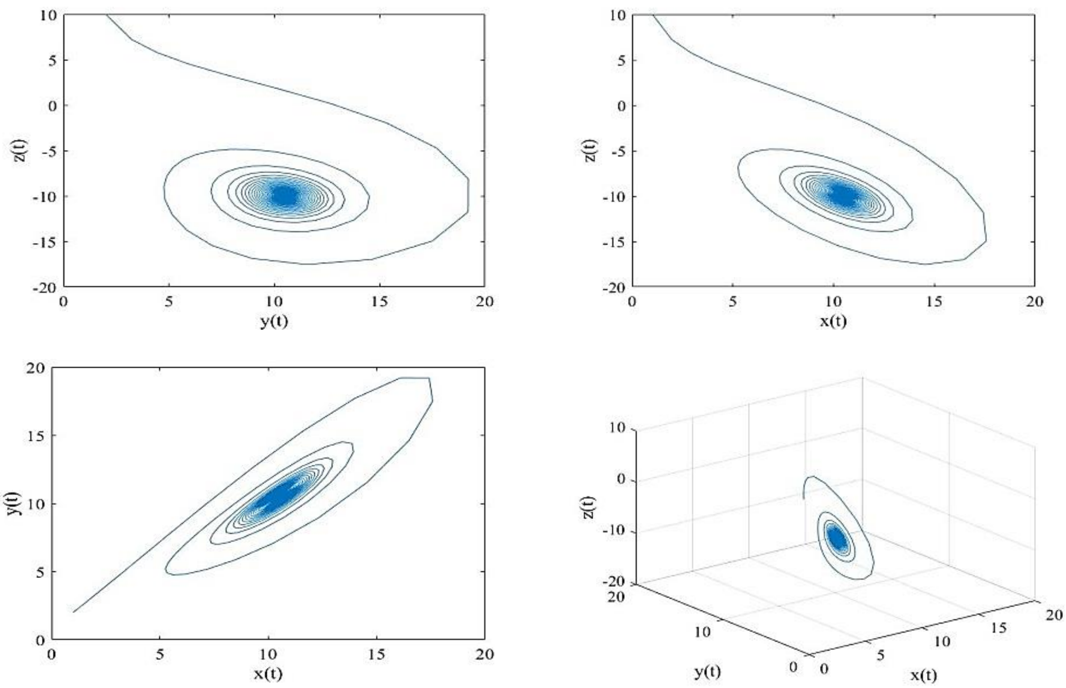


Fig. 3. *The system's phase diagram for the order* $\alpha(t) = \tanh(\frac{\pi}{2} + t)$.



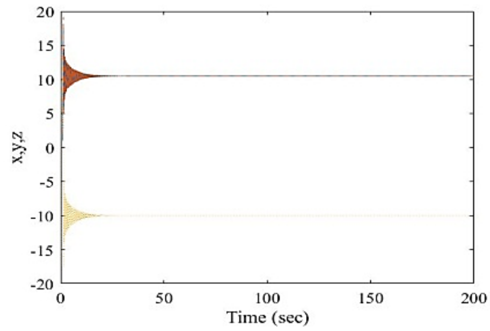


Fig. 4. *Proposed robust controller of the system for the order* $\alpha(t) = \tanh(\frac{\pi}{2} + t)$.

7. Conclusion

A variable order butterfly- shaped chaotic attractor fractional chaotic system is considered by a numerical solution based on the Adams method. Numerical solutions are successfully obtained and the method is demonstrated to operate accurately and powerfully. Numerical examples with different Atangana-Baleanu-Caputo variable- order were given to prove that the method is effective. The robust control of this system is investigated and it is stated that the control has a more flexible and general structure which was one of the motivations of work presented here.

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