J. Appl. Res. Ind. Eng. Vol. 8, No. 1 (2021) 90–103

Journal of Applied Research on Industrial Engineering
www.journal-aprie.com

Numerical Approximation for the Fractional Advection-Diffusion Equation Using a High Order Difference Scheme

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PAPER INFO

Chronicle:
Received: 20 July 2020
Reviewed: 05 September 2020
Revised: 04 February 2021
Accepted: 02 March 2021

Keywords:
Fractional Advection-Diffusion.
Riemann-Liouville.
Stability Analysis.

ABSTRACT

In this paper, a one-dimensional fractional advection-diffusion equation is considered. First, we propose a numerical approximation of the Riemann-Liouville fractional derivative which is fourth-order accurate, then a numerical method for the fractional advection-diffusion equation using a high order finite difference scheme is presented. It is proved that the scheme is convergent. The stability analysis of numerical solutions is also discussed. The method is applied in several examples and the accuracy of the method is tested in terms of L_∞ error norm. Furthermore, the numerical results have been compared with some other methods.

1. Introduction

Fractional calculus is a useful mathematical tool for applied sciences. The fractional advection–diffusion equations provide an adequate and accurate description of the movement of solute in an aquifer. However, there are major obstacles that restrict their applications. From a modeling viewpoint, the fractional advection diffusion equation has been presented as a more suitable model for many problems that appear in different fields, such as engineering, physics, chemistry and hydrology.

Fractional differential equation are generalizations of classical differential equations of integer order that have recently proved to be valuable tools for the modelling of many physical phenomena and have been the focus of many studies due to their frequent appearances in various applications, such as physics, biology, finance and fractional dynamics, engineering, signal processing, and control theory [15].

Mahboob Dana, Z., Saberi Najafi, H., & Refahi Sheikhani, A. H. (2021). Numerical approximation for the fractional advection-diffusion equation using a high order difference scheme. *Journal of applied research on industrial engineering, 8*(1), 90-103.

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10.22105/jarie.2021.240340.1183

The fractional advection diffusion equation was given by

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + D\left(\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial (-x)^{\alpha}}\right), \quad (1)$$

Where u is the concentration, V is the average velocity, x is the spatial coordinate, t is the time, D is the diffusion coefficient, α is the order of the fractional differentiation with $1<\alpha\le 2$. The fractional advection diffusion equation was later generalized by Benson et al. [16] to include the parameter β, given by

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + D\left(\frac{1}{2}+\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + D\left(\frac{1}{2}-\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial (-x)^{\alpha}}, \quad (2)$$

for $-1\le\beta\le 0$, the transition probability is skewed backward, while for $0\le\beta\le 1$ the transition probability is skewed forward. For $\beta = 0$, we obtain the model presented in [17], which can be expressed as follows

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x}\nabla_{\alpha}^{\beta} u(x,t) + p(x,t) \quad (3)$$

where the fractional operator is given by

$$\nabla_{\alpha}^{\beta} u(x,t) = \left(\frac{1}{2}+\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \left(\frac{1}{2}-\frac{\beta}{2}\right)\frac{\partial^{\alpha} u(x,t)}{\partial (-x)^{\alpha}}, \quad (4)$$

with initial condition

$u(x,0) = f(x), x \in \Re$

The Riemann–Liouville fractional derivatives of order α, for $x\in[a,b], -\infty\le a<b\le\infty$, are defined by

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^{n}}{\partial x^{n}}\int_{a}^{x} u(\xi,t)(x-\xi)^{n-\alpha-1}d\xi, n-1<\alpha<n. \quad (5)$$

$$\frac{\partial^{\alpha} u}{\partial (-x)^{\alpha}}(x,t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\frac{\partial^{n}}{\partial x^{n}}\int_{x}^{b} u(\xi,t)(\xi-x)^{n-\alpha-1}d\xi, n-1<\alpha<n. \quad (6)$$

A meshless method of *Eq. (1)* has been discussed by Mardani et al. [1] and Tayebi et al. [2]. A functional variable method is given in Aminikhah et al. [4]. An first integral method for the solution of fractional differential equations is given in [5]. Lattice Boltzmann method [3]. And finite element methods for linear and nonlinear diffusion problems [7], [8], [9], [12], [13], and [14]. In this paper, by using a Lax–Wendroff-type time discretization procedure, we develop an explicit numerical method which is second order in time and space for fractional advection diffusion problems with source terms in unbounded and bounded domains with homogeneous boundary conditions. Since the numerical method is explicit, it is a more cost-effective method than the implicit schemes. Additionally, explicit methods are better tools for problems wherein advection plays an important role. The classical Lax–Wendroff method was derived for hyperbolic equations [18] and afterwards was extended for advection diffusion equations. This method uses a small stencil in time and also uses the original differential equation extensively, that is, the discretization procedure converts time derivatives in space derivatives. The layout of this paper is as follows. In Section 2, we will illustrate construction of the scheme and in Section 3 we describe the numerical method. In Section 4, we study the stability analysis. In the Section 5, we will prove the convergence of difference scheme analysis. The Section 6 includes some numerical tests which confirm the fourth-order convergence of the numerical method. A summary is given at the end of the paper in Section 7.

2. Construction of the Scheme

we present a new numerical approximation that this approximation is fourth-order accurate. Consider first the left derivative, that is,

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}}(x,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} u(\xi,t)(x-\xi)^{1-\alpha}d\xi, 1<\alpha<2. \quad (7)$$

We define the mesh points $x_j = j\Delta x, j \in Z$ where $h=\Delta x$ denotes the uniform space step. For a fixed time t, let us denote

$$I_{\alpha}(x)=\int_{-\infty}^{x} u(\xi,t)(x-\xi)^{1-\alpha}d\xi. \quad (8)$$

First, we do the following approximation at x_j

$$\frac{\partial^2}{\partial x^2} I_{\alpha}(x_j) \approx \frac{-I_{\alpha}(x_{j+2})+16I_{\alpha}(x_{j+1})-30I_{\alpha}(x_j)+16I_{\alpha}(x_{j-1})-I_{\alpha}(x_{j-2})}{12h^2}, \quad (9)$$

and $i_{\alpha}(x)$ defined by

$$i_{\alpha}(x_{j+1})=\int_{-\infty}^{x_{j+1}} s_{j+1}(\xi)(x_{j+1}-\xi)^{1-\alpha}d\xi, \quad (10)$$

The spline $s_{j+1}(\xi)$ interpolates the points $\{(x_k,t): k \le j+1\}$ and is of the form [15]

$$s_{j+1}(\xi)=\sum_{k=-\infty}^{j+1} u(x_k,t)s_{j+1,k}(\xi). \quad (11)$$

In each interval $[x_{k-2},x_{k+2}]$ for $k \le j+1$ we have

$$s_{j+1,k}(\xi)=\begin{cases}\dfrac{\xi-x_{k-2}}{x_{k-1}-x_{k-2}} & x_{k-2}\le\xi\le x_{k-1}\\ \dfrac{\xi-x_{k-1}}{x_k-x_{k-1}} & x_{k-1}\le\xi\le x_k\\ \dfrac{\xi-x_k}{x_{k+1}-x_k} & x_k\le\xi\le x_{k+1}\\ \dfrac{x_{k+1}-\xi}{x_{k+2}-x_{k+1}} & x_{k+1}\le\xi\le x_{k+2}\\ 0 & \text{otherwise}\end{cases}. \tag{12}$$

And for $k=j+1$,

$$s_{j+1,j+1}=\begin{cases}\dfrac{\xi-x_{j+1}}{x_{j+2}-x_{j+1}} & x_j\le\xi\le x_{j+1}\\ 0 & \text{otherwise}\end{cases}. \tag{13}$$

From *Eqs. (10)* and *(11)*, we have

$$i_\alpha(x_{j+1})=\sum_{k=-\infty}^{j+1}u(x_k,t)\int_{x_{k-2}}^{x_{k+2}}s_{j+1,k}(\xi)(x_{j+1}-\xi)^{1-\alpha}d\xi. \tag{14}$$

Therefore,

$$\int_{x_{k-2}}^{x_{k+2}}s_{j+1,k}(\xi)(x_{j+1}-\xi)^{1-\alpha}d\xi=\int_{x_{k-2}}^{x_{k-1}}\frac{\xi-x_{k-2}}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+\int_{x_{k-1}}^{x_k}\frac{\xi-x_{k-1}}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+$$
$$\int_{x_k}^{x_{k+1}}\frac{\xi-x_k}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi+\int_{x_{k+1}}^{x_{k+2}}\frac{x_{k+1}-\xi}{h}(x_{j+1}-\xi)^{1-\alpha}d\xi=\frac{\Delta x^{2-\alpha}}{(2-\alpha)(3-\alpha)}a_{j+1,k}. \tag{15}$$

Where

$$a_{j+1,k}=\begin{cases}-(j-k-2)^{3-\alpha}+16(j-k-1)^{3-\alpha}-30(j-k)^{3-\alpha}+16(j-k+1)^{3-\alpha}-(j-k+2)^{3-\alpha} & k\le j\\ 1 & k=j+1\end{cases}. \tag{16}$$

Therefore,

$$i_\alpha(x_{j+1})=\frac{\Delta x^{2-\alpha}}{(2-\alpha)(3-\alpha)}\sum_{k=-\infty}^{j+1}u(x_k,t)a_{j+1,k}. \tag{17}$$

And an approximation for $\frac{\partial^2}{\partial x^2}I_\alpha(x_{j+1})$, is given by

$$\frac{-I_\alpha(x_{j+2})+16I_\alpha(x_{j+1})-30I_\alpha(x_j)+16I_\alpha(x_{j-1})-I_\alpha(x_{j-2})}{12h^2}.$$

That is

$$\frac{\Delta x^{-\alpha}}{12(2-\alpha)(3-\alpha)}\left[\begin{array}{l}\sum_{k=-\infty}^{j+2} u(x_k,t)a_{j+2,k}+16\sum_{k=-\infty}^{j+1} u(x_k,t)a_{j+1,k}-30\sum_{k=-\infty}^{j} u(x_k,t)a_{j,k}+\\ 16\sum_{k=-\infty}^{j-1} u(x_k,t)a_{j-1,k}-\sum_{k=-\infty}^{j-2} u(x_k,t)a_{j-2,k}\end{array}\right]. \quad (18)$$

We assume there are approximation U_j^n to the values $u(x_j,t_n)$, where $t_n=n\Delta t, n\geq 0$ and we define the fractional operator as

$$\delta_\alpha U_j^n=\frac{1}{12\Gamma(4-\alpha)}\sum_{k=-\infty}^{j+2} q_{j,k}U_k^n. \quad (19)$$

Where

$$\begin{array}{ll} q_{j,k}=-a_{j+2,k}+16a_{j+1,k}-30a_{j,k}+16a_{j-1,k}-a_{j-2,k} & k\leq j-2\\ q_{j,j-1}=-a_{j+2,j-1}+16a_{j+1,j-1}-30a_{j,j-1}+16a_{j-1,j-1} & \\ q_{j,j}=-a_{j+2,j}+16a_{j+1,j}-30a_{j,j} & \\ q_{j,j+1}=-a_{j+2,j+1}+16a_{j+1,j+1} & \\ q_{j,j+2}=-a_{j+2,j+2}. & \end{array} \quad (20)$$

Therefore, an approximation of *Eq. (7)* can be given by $\frac{\delta_\alpha U_j^n}{\Delta x^\alpha}$.We can also write the fractional operator *Eq. (19)* as

$$\delta_\alpha U_j^n=\frac{1}{12\Gamma(4-\alpha)}\sum_{m=-2}^{\infty} q_{j,j-m}U_{j-m}^n, \quad (21)$$

we define,

$$a_m=\begin{cases}-(m+2)^{3-\alpha}+16(m+1)^{3-\alpha}-30(m)^{3-\alpha}+16(m-1)^{3-\alpha}-(m-2)^{3-\alpha} & m\geq 2\\ 1 & m=0\end{cases}. \quad (22)$$

And

$$q_m=\begin{cases}-a_{m+2}+16a_{m+1}-30a_m+16a_{m-1}-a_{m-2} & m\geq 2\\ -a_3+16a_2-30a_1+16a_0 & m=1\\ -a_2+16a_1-30a_0 & m=0\\ -a_1+16a_0 & m=-1\\ -a_0 & m=-2\end{cases}, \quad (23)$$

we have

$$\delta_\alpha U_j^n=\frac{1}{12\Gamma(4-\alpha)}\sum_{m=-2}^{\infty} q_m U_{j-m}^n. \quad (24)$$

And similarly in the interval $[x,\infty)$

$$\delta'_\alpha U_j^n = \frac{1}{12\Gamma(4-\alpha)} \sum_{m=-2}^{\infty} q_m U_{j+m}^n. \tag{25}$$

3. Numerical Method

To derive a finite difference scheme, we suppose there are approximations $U^n = U_j^n$. to the values $u(x_j, t_n)$ at the mesh points $x_j = j\Delta x, h = \Delta x, j \in Z$ and $t_n = n\Delta t, n \geq 0$.

We assume $s = \frac{V\Delta t}{2\Delta x}$ and $\mu = \frac{\Delta t D}{\Delta x^\alpha}$,

we expand u about time level n, that is, $t = n\Delta t$ to obtain

$$u(x,t_{n+1}) - u(x,t_n) = \Delta t \frac{\partial u}{\partial t}(x,t_n) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(x,t_n) + O(\Delta t^3). \tag{26}$$

Then, from *Eq. (26)* we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) = -V(x,t)\frac{\partial^2 u}{\partial x \partial t}(x,t) + d(x,t)\nabla_\alpha^\beta\left(\frac{\partial u}{\partial t}(x,t)\right) + p_t(x,t). \tag{27}$$

And

$$\frac{\partial^2 u}{\partial t^2}(x,t) \approx V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t) - V(x,t)p_x(x,t) + p_t(x,t). \tag{28}$$

Inserting *Eqs.(3)* and *(29)* into *Eq.(26)* gives

$$\begin{aligned} u(x,t_{n+1}) &\approx u(x,t_n) + \Delta t\left(-V(x,t)\frac{\partial u}{\partial x}(x,t_n) + d(x,t)\nabla_\beta^\alpha u(x,t_n) + p(x,t_n)\right) + \\ &\frac{\Delta t^2}{2}\left(V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t_n) - V(x,t)p_x(x,t) + p_t(x,t)\right). \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned} u(x,t_{n+1}) &\approx u(x,t_n) - V(x,t)\Delta t\frac{\partial u}{\partial x}(x,t_n) + \Delta t d(x,t)\nabla_\beta^\alpha u(x,t_n) + \\ &\frac{\Delta t^2}{2}V^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t_n) + \Delta t\left(p\ (x,t) + \frac{\Delta t}{2}\left(-V(x,t)p_x(x,t) + p_t(x,t)\right)\right). \end{aligned} \tag{30}$$

We define the following operators,

$$\frac{\partial u}{\partial x}(x,t_n)=\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{12h}+o(\Delta x^4),$$
$$\frac{\partial^2 u}{\partial x^2}(x,t_n)=\frac{-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n}{12h^2}+o(\Delta x^4). \quad (31)$$

And the fractional operator

$$\delta_\beta^\alpha u(x,t_n)=\left(\frac{1}{2}+\frac{\beta}{2}\right)\delta_\alpha u(x,t_n)+\left(\frac{1}{2}-\frac{\beta}{2}\right)\delta'_\alpha u(x,t_n). \quad (32)$$

We introduce the following notations

$$\delta U_j^n=-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n$$
$$\delta^2 U_j^n=-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n. \quad (33)$$

We obtain

$$\overline{p}_j^n=p_j^n+\frac{\Delta t}{2}\left(-V(x,t)\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{12h}+\frac{U_j^{n+1}-U_j^n}{\Delta t}\right).$$

Where

$$p_j^n=p(x_j,t_n). \quad (34)$$

Therefore, from *Eq. (30)* we have

$$U_j^{n+1}=U_j^n-\frac{s}{6}\delta U_j^n+\mu\delta_\beta^\alpha U_j^n+\frac{1}{6}s^2\delta^2 U_j^n+\Delta t\overline{p}_j^n. \quad (35)$$

4. Stability

If u_j^n is the exact solution $u(x_j,t_n)$, let U_j^n be a perturbation of u_j^n. The perturbation error

$$e_j^n=U_j^n-u_j^n,$$

will be propagated forward in time according to the equation

$$e_j^{n+1}=e_j^n-\frac{s}{6}\delta e_j^n+\mu\delta_\beta^\alpha e_j^n+\frac{1}{6}s^2\delta^2 e_j^n. \quad (36)$$

Theorem 1. For $\beta=0$, the numerical method is von Neumann stable if, and only if,

$$\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\le-\frac{1}{12}. \quad (37)$$

Proof. We have

$$G(\theta)=1+\frac{s}{6}i(-4\sin^2\theta+16\sin\theta)+\frac{s^2}{6}(-4\cos^2\theta+32\cos\theta-30)+$$
$$\frac{\mu}{24\Gamma(4-\alpha)}\sum_{m=-2}^{\infty}q_m\cos(m\theta).$$

That is

$$|G(\theta)|^2=\left[1+\frac{s^2}{6}(-4\cos^2\theta+32\cos\theta-30)+\frac{\mu}{24\Gamma(4-\alpha)}\sum_{m=-2}^{\infty}q_m\cos(m\theta)\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right],^2$$

if we have *Condition (37)* then $|G(\theta)|\le 1$, for all θ. We have that

$$\sum_{m=-2}^{\infty}q_m\cos(m\theta)\le(q_{-2}+q_2)\cos 2\theta+(q_{-1}+q_1)\cos\theta+q_0+\sum_{m=3}^{\infty}q_m=$$
$$2(q_{-2}+q_2)(\cos^2\theta-1)+(q_{-1}+q_1)(\cos\theta-1).$$

Therefore,

$$|G(\theta)|^2\le\left[\begin{array}{l}1-\frac{s^2}{3}(2\cos^2\theta-16\cos\theta+15)+\frac{\mu}{12\Gamma(4-\alpha)}(\cos^2\theta-1)(q_{-2}+q_2)+\\ \frac{\mu}{24\Gamma(4-\alpha)}(\cos\theta-1)(q_{-1}+q_1)\end{array}\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right]^2.$$

Then,

$$|G(\theta)|^2\le$$
$$\left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)\right]^2$$
$$-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right].^2$$

For $K=\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)$.

It follows

$$|G(\theta)|^2 \le \left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)K\right]^2-\left[\frac{s}{6}(-4\sin^2\theta+16\sin\theta)\right]^2$$
$$\le \left[1-\frac{s^2}{3}(2\cos^2\theta-1)+(\cos\theta-1)K\right]^2=1+\frac{s^4}{9}(2\cos^2\theta-1)-\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1).$$

Note that

$$\left[\sum_{m=3}^{\infty}(-1)^m q_m+q_0\right]\le 0.$$

Then

$$K=\left(\frac{16}{3}s^2+\frac{\mu}{12\Gamma(4-\alpha)}(\cos\theta+1)(q_{-2}+q_2)+\frac{\mu}{24\Gamma(4-\alpha)}(q_{-1}+q_1)\right)\le$$
$$\frac{16}{3}+\frac{\mu}{6\Gamma(4-\alpha)}\left[\sum_{m=3}^{\infty}(-1)^m q_m+q_0\right]\le\frac{16}{3}.$$

And

$$|G(\theta)|^2\le 1+\frac{s^4}{9}(2\cos^2\theta-1)-\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1).$$

We know that $|G(\theta)|\le 1$ is stability condition; therefore, it is necessary that

$$\frac{2Ks^2}{3}(\cos\theta-1)(2\cos^2\theta-1)\ge\frac{s^4}{9}(2\cos^2\theta-1).$$

Therefore, for $K\le-\frac{1}{12}$, we have $|G(\theta)|^2\le 1$, for all θ .

5. Convergence Analysis

In this section we analyze the convergence of the numerical method using the framework of consistency and stability. We have the global error given by $e^n=u^n-U^n$, where u^n and U^n are respectively exact and approximate solutions. The truncation error at each discrete point x_j , is given by

$$E_j^n=\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{V}{6}\frac{-U_{j+2}^n+8U_{j+1}^n-8U_{j-1}^n+U_{j-2}^n}{\Delta t}-$$
$$\frac{V^2\Delta t}{6}\frac{-U_{j+2}^n+16U_{j+1}^n-30U_j^n+16U_{j-1}^n-U_{j-2}^n}{h^2}-\frac{D}{2\Delta x^\alpha}\delta_\beta^\alpha u_j^n=$$
$$\left(\frac{\partial u}{\partial t}\right)_j^n+\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_j^n+O(\Delta t^2)+V\left(\frac{\partial u}{\partial x}\right)_j^n+O(\Delta x^4)-$$
$$\frac{V^2\Delta t}{6}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n+O(\Delta x^4)-D(\nabla_\beta^\alpha u)_j^n+O(\Delta x^4).$$

Therefore,

$$E_j^n = \left(\frac{\partial u}{\partial t}\right)_j^n + O(\Delta t^2) + V\left(\frac{\partial u}{\partial x}\right)_j^n + O(\Delta x^4) - \frac{V^2 \Delta t}{6}\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n + O(\Delta x^4) - D(\nabla_\beta^\alpha u)_j^n + O(\Delta x^4).$$

6. Numerical Experiments

In this section we consider examples that are solved by our presented method in previous sections. In order to illustrate the accuracy of the method, we used the error norm L_∞ which is defined as follows

$$L_\infty = \max_j \left|U(x_j, t) - u(x_j, t\ )\right|.$$

And we compare our results with the results in [10].

<u>*Example 1.*</u> We assume $\beta = 1$ in *Eq. (3)*, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + D\nabla_\alpha^\beta u(x,t) + p(x,t). \tag{38}$$

In the domain $0 \le x \le 1$, we assume the problem has initial condition $u(x,0) = x^4$ and boundary conditions $u(0,t) = 0, u(1,t) = e^{-t}$. Let $V = 0.2$ $D = \frac{\Gamma(5-\alpha)}{24}$ And $p(x,t) = e^{-t}x^3(4V - x - x^{1-\alpha})$.

The exact solution is given by $u(x,t) = e^{-t}x^4$.

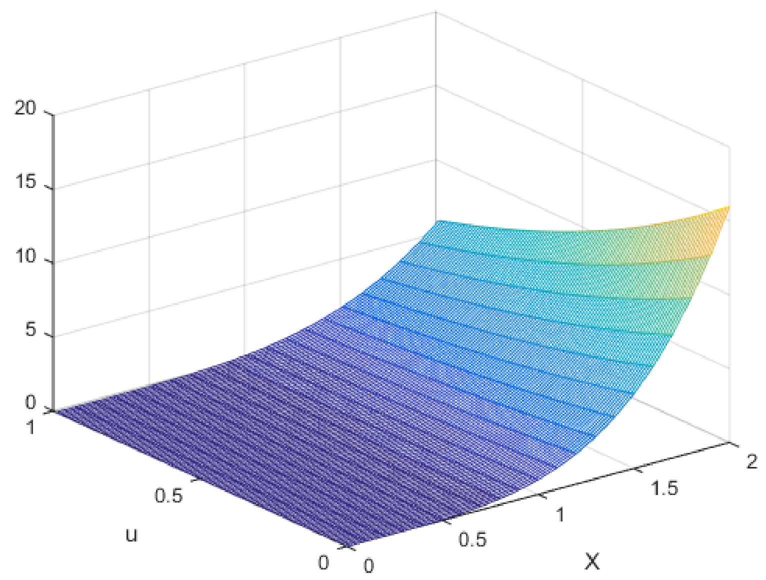

Fig. 1. *Shows the exact solution of Eq. (38) when* $\beta = 1$.

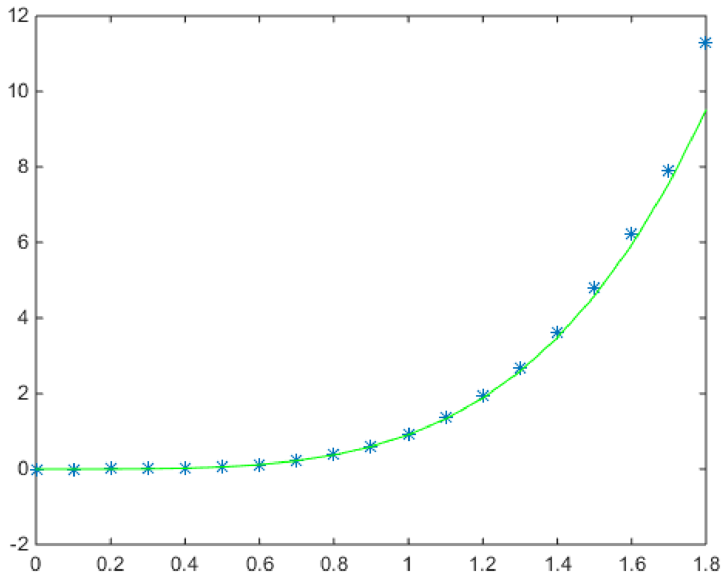

Fig. 2. *Comparison of the exact and numerical solution with* $\alpha = 1.2, \beta = 1$.

Table 1. *The computational results for example 1 by our method and method in [10] and [3].*

		$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 2$
Our method $\Delta x = 0.1$	L_∞	0.2376×10^{-7}	0.17609×10^{-7}	0.2151×10^{-7}	0.16705×10^{-7}
$\Delta x = 0.01$	L_∞	0.2256×10^{-9}	0.1741×10^{-9}	0.2004×10^{-9}	0.1079×10^{-9}
Method in [10] $\Delta x = 0.1$	L_∞	0.4603×10^{-3}	0.3208×10^{-3}	0.4453×10^{-3}	0.1239×10^{-2}
$\Delta x = 0.01$	L_∞	0.5667×10^{-4}	0.4444×10^{-4}	0.4864×10^{-4}	0.1095×10^{-4}
Method in [3] $\Delta x = 0.1$	L_∞	0.5512×10^{-5}	0.3209×10^{-5}	0.1729×10^{-6} 0.7412×10^{-5}	0.1782×10^{-5}
$\Delta x = 0.01$	L_∞	0.1920×10^{-7}	0.6724×10^{-7}	0.4132×10^{-7}	

Table 1 shows the errors of our proposed method with values of h at different final times using L_∞. Numerical results of this table confirm that the method has fourth-order of accuracy in temporal and spatial components, respectively. Comparison of this method to other methods, [10] and [3], confirms the efficiency and high accuracy of our proposed method.

The second example considers *Eq. (3)* for $\beta = 0$, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + \frac{D}{2}\left(\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial (-x)^\alpha}\right) + p(x,t). \quad (39)$$

Example 2. The second example considers *Eq. (3)* for $\beta = 0$, that is, we have the equation

$$\frac{\partial u(x,t)}{\partial t} = -V\frac{\partial u(x,t)}{\partial x} + \frac{D}{2}\left(\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial (-x)^{\alpha}}\right) + p(x,t).$$

In the domain $0 \le x \le 2$. We assume the initial condition is $u(x,0) = 4x^2(2-x)^2$ and the boundary conditions are $u(0,t) = 0, u(2,t) = 0$.

Let $V = 0.05$ $D = \frac{\Gamma(5-\alpha)}{2}$ and

$$p(x,t) = 4e^{-t}(-x^2(2-x)^2 + 4Vx(x^2-3x+2) - x^{2-\alpha}A(x,\alpha) - (2-x)^{2-\alpha}B(x,\alpha)).$$

Where

$$A(x,\alpha) = 2\alpha(\alpha-1) - 6\alpha(2-x) + 6(2-x)^2$$
$$B(x,\alpha) = 2\alpha(\alpha-1) - 6\alpha x + 6x^2.$$

The exact solution is given by $u(x,t) = 4e^{-t}x^2(2-x)^2$.

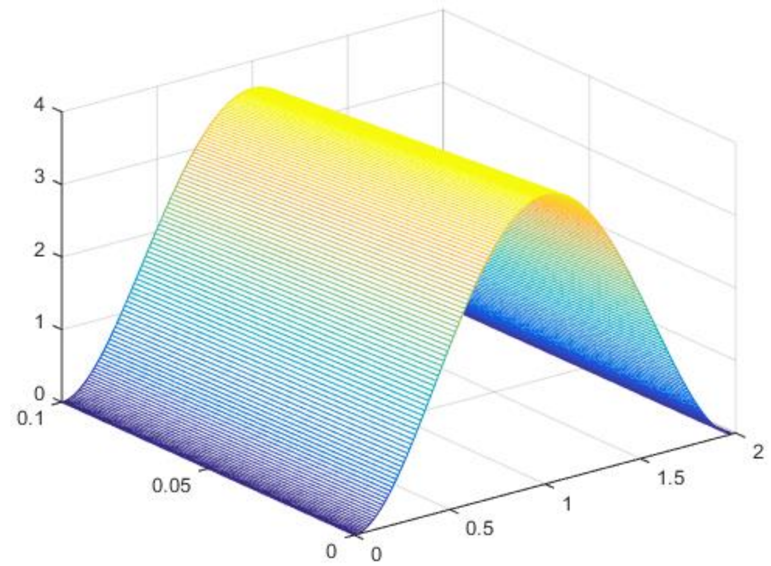

Fig. 3. *Shows the numerical approximation of Eq. (39) when* $\beta = 0$.

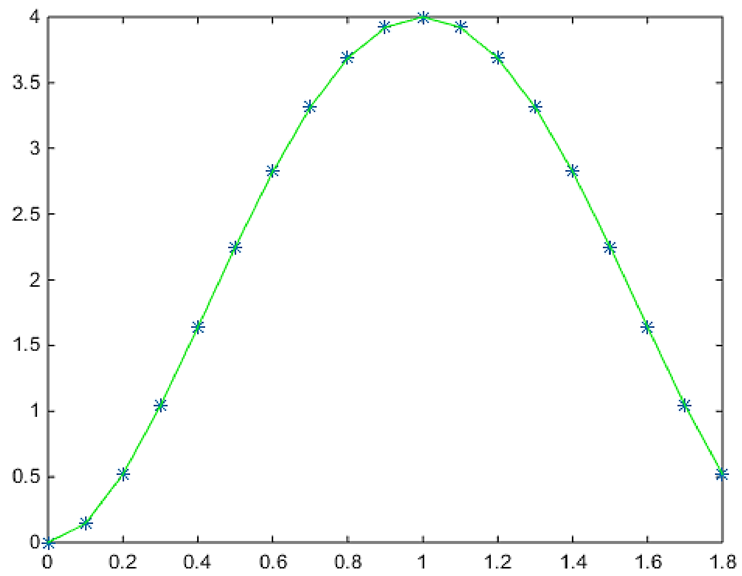

Fig. 4. *Comparison of the exact and numerical solution with* $\beta = 0, \Delta x = 0.1$.

Table 2. *The computational results for example 2.*

			$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 2$
Our method	$\Delta x = 0.1$	L_∞	0.7823×10^{-5}	0.1240×10^{-5}	0.1782×10^{-5}	0.4512×10^{-5}
	$\Delta x = 0.01$	L_∞	0.2361×10^{-8}	0.1421×10^{-8}	0.2344×10^{-8}	0.1761×10^{-8}
Method [6]	$\Delta x = 0.1$	L_∞	0.3216×10^{-3}	0.14213×10^{-3}	0.1214×10^{-3}	0.2325×10^{-3}
	$\Delta x = 0.01$	L_∞	0.7451×10^{-5}	0.4617×10^{-5}	0.1238×10^{-5}	0.2113×10^{-5}

According to *Table 2* the results indicate that our supposed scheme has a high accuracy and shows that errors are very small. our method is conditionally stable, consistent and convergent, which is fourth-order accurate with respect to the space step and second – order accurate to the time step.

Therefore, our method is more convenient than the [6], which is second order accurate with respect to the space step.

7. Conclusion

In this work, we have applied a high order finite difference method for the fractional advection-diffusion equation. Furthermore, we proved that this scheme is stable and convergent. Numerical results of the above tables confirm that the method has fourth-order of accuracy in temporal and spatial components, and the errors are very small. This scheme is an accurate and efficient approach for the solution of such types of nonlinear partial differential equations, we suggest to use this method for solving nonlinear equations [6] and [11]. We also see that the method presented produces very good results compared with the second order and fourth order methods proposed in [10] and [3]. Comparison of this method to other methods confirms the efficiency and high accuracy of our proposed method.

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