# A Method Based on the SCT to Solve the Inverse Problem for Heat Equation 

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#### Abstract

In this article, a mathematical model of the inverse problem is considered. Based on this model a formulation of inverse problem for heat equation is proposed. Shifted Chebyshev Tau (SCT) method is suggested to solve the inverse problem. The aim of this determined effort is to identify unknown function and unknown control parameter of the mathematical model. In order to achieve highly accurate solution to this problem, the operational matrix of shifted Chebyshev polynomials is investigated in conjunction with tau scheme. To demonstrate the validity and applicability of the developed scheme, numerical example is presented.


Keywords: Inverse problem, Heat equation, Operational matrix, SCT method.

## 1 | Introduction

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In this paper, a meticulous presentation is begun with an inverse problem arising in the heat equation, which aims to determine the function $u(x, t)$ and control parameter $P(t)$ in the following form [1]

$$
\begin{equation*}
u_{t}(x, t)=u_{x x}(x, t)+p(t) u(x, t)+q(x, t) \tag{1}
\end{equation*}
$$

for $(x, t) \in(0, L) \times(0, \tau]$, where $x$ and $t$ represent space and time variables, respectively;
with initial condition

$$
\begin{equation*}
\mathrm{u}(\mathrm{x}, 0)=\mathrm{f}(\mathrm{x}), \quad 0<\mathrm{x}<\mathrm{L} \tag{2}
\end{equation*}
$$

and boundary conditions

$$
\begin{align*}
& \alpha_{1}(\mathrm{t}) \mathrm{u}_{\mathrm{x}}(0, \mathrm{t})+\beta_{1}(\mathrm{t}) \mathrm{u}(0, \mathrm{t})+\gamma_{1}(\mathrm{t}) \mathrm{u}(\mathrm{~L}, \mathrm{t})=\mathrm{g}_{1}(\mathrm{t}), 0<\mathrm{t} \leq \tau,  \tag{3}\\
& \alpha_{2}(\mathrm{t}) \mathrm{u}_{\mathrm{x}}(\mathrm{~L}, \mathrm{t})+\beta_{2}(\mathrm{t}) \mathrm{u}(0, \mathrm{t})+\gamma_{2}(\mathrm{t}) \mathrm{u}(\mathrm{~L}, \mathrm{t})=\mathrm{g}_{2}(\mathrm{t}), 0<\mathrm{t} \leq \tau, \tag{4}
\end{align*}
$$

and the energy condition

$$
\begin{equation*}
\int_{0}^{s(t)} u(x, t) d x=E(t), \quad 0<t \leq \tau, \quad 0<s(t) \leq L \tag{5}
\end{equation*}
$$

where $q(x, t), f(x), E(t), \alpha_{i}(t), \beta_{i}(t), \gamma_{i}(t), g_{i}(t), i=1,2$ are given functions.

Determination of an unknown control parameter is one of the hottest topics in inverse problems. There are many papers studying this type of equation [1]-[8]. Previously mentioned equation arise in many fields of science and engineering such as chemical diffusion, heat conduction processes, population dynamics, thermoelasticity, medical science, electrochemistry and control theory [9]-[15]. For example, microwave heat process used in various applications in industry, can be seen in ceramics and in food processing where the external energy is supplied to the target at a controlled level by the microwave-generating equipment. This can correspond to source term $p(t) u(x, t)$ in Eq. (1), where $p(t)$ is proportional to power of external energy source and $u(x, t)$ is local conversion rate of microwave energy [12].

The energy Condition (1) is applied when the value of the control parameter $p(t)$ cannot be obtained using classical boundary conditions (which is divided into three phases: 1) neumann, 2) dirichlet, and 3) robin). Such type of condition can model various physical phenomena in context of heat transfer, life science and etc. [1], [2], [5], [6], [16], [17], [18]. The existence, uniqueness and continuous dependence of the solution upon the date for this problem are demonstrated in [1] under the following assumptions:

$$
\begin{aligned}
& q \in C^{a, a / 2}([0, L] \times[0, \tau]), \text { for } 0<a \leq 1, f \in C^{1}[0, L], g_{1}, g_{2} \in C[0, \tau], \\
& E(0)=\int_{0}^{s(0)} f(x) d x>0, \quad E(t)>0, s(t), E(t) \in C^{1}[0, \tau], \quad \alpha_{i}, \beta_{i}, \gamma_{i} \in C[0, \tau], \\
& i=1,2, \text { and } \\
& \alpha_{1}(0) f_{x}(0)+\beta_{1}(0) f(0)+\gamma_{1}(0) f(L)=g_{1}(0) \\
& \alpha_{2}(0) f_{x}(L)+\beta_{2}(0) f(0)+\gamma_{2}(0) f(L)=g_{2}(0) .
\end{aligned}
$$

Approximation and numerical solution of an inverse heat equation by control parameter are discussed in many papers, such as Boundary element method [8], finite volume element method [19], Generalized Fourier method [12], radial basis function collocation method [20], collocation method [21], Sinccollocation method [17] and [6] third order compact Runge-Kutta method [8] and other mthods [3], [7], [21]-[27], must be used. This paper presents a simple and efficient algorithm for finding an approximate solution of Eq. (1) under the Conditions (2) to (4) and the energy Condition (5). Instead, an algorithm which is called Shifted Chebyshev Tau (SCT), is proposed.

The main aim of this research is to use SCT method to solve an inverse heat Eq. (5). Shifted Chebyshev polynomials of the first kind are put into practice to approximate the solution of the equation as a base of the tau method which is based on the shifted Chebyshev operational matrices of derivative and integration. The main advantage of this method is based upon reducing the PDE into a system of algebraic equation in the coefficient expansion of the solution. Numerical example, which confirm the accuracy of this method, is presented.

The presentation of this paper is as follows: a pair of transformations is brought to change the structure of the Eqs. (1) to (5), then highlighting some necessary definitions and matrix formulation of Shifted Chebyshev polynomials, and construct its operational matrices of derivative and integral. In Section 3, the presented SCT method is used to find the approximate solution of the problem. As a result, a set of algebraic equations is formed and the solution of the considered problem is introduced. In Section 4, we
discussed an error bound. Numerical results in Section 5 is given to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 6.

## 2 | Preparation and Foundation

At first, we that the pair of transformations constructed in follow:

$$
\begin{align*}
& r(t)=\exp \left(-\int_{0}^{t} p(s) d s\right),  \tag{6}\\
& v(x, t)=r(t) u(x, t) . \tag{7}
\end{align*}
$$

The Problems (1) to (5) will become [5]:

$$
\begin{equation*}
v_{t}=v_{x x}+r(t) q(x, t), \quad 0<x<L, 0<t \leq \tau \tag{8}
\end{equation*}
$$

Subject to

$$
\begin{align*}
& v(x, 0)=f(x), \quad 0<x<L  \tag{9}\\
& \alpha_{1}(t) v_{x}(0, t)+\beta_{1}(t) v(0, t)+\gamma_{1}(t) v(L, t)=r(t) g_{1}(t), \quad 0<t \leq \tau,  \tag{10}\\
& \alpha_{2}(t) v_{x}(L, t)+\beta_{2}(t) v(0, t)+\gamma_{2}(t) v(L, t)=r(t) g_{2}(t), \quad 0<t \leq \tau, \tag{11}
\end{align*}
$$

and

$$
\begin{equation*}
\int_{0}^{s(t)} v(x, t) d x=r(t) E(t), \quad 0<t \leq \tau, \quad 0<s(t)<L . \tag{12}
\end{equation*}
$$

Obviously, if we have $v(x, t)$ and $r(t)$ then by using Eqs. (8) to (12), $u(x, t)$ and $p(t)$ can be found as:

$$
\begin{array}{ll}
u(x, t)=\frac{v(x, t)}{r(t)}, & 0<x<L, 0<t \leq \tau, \\
p(t)=-\frac{r^{\prime}(t)}{r(t)}, & 0<t \leq \tau . \tag{14}
\end{array}
$$

In this transformation, the source parameter disappeared, so we can solve the Eqs. (8) to (12).

## 2.1 | Basic Definitions and Matrix Formulation

In this section, some fundamental definitions are given and to introduce the necessary notation, also matrix formulation of Shifted Chebyshev polynomial of the first kind which will be used throughout the paper.

The shifted Chebyshev polynomials are generated from the following three-term recurrence relation:

$$
\begin{align*}
& T_{L, 0}(x)=1, T_{L, 1}(x)=\frac{2 x}{L}-1 \\
& T_{L, j}(x)=2\left(\frac{2 x}{L}-1\right) T_{L, j-1}(x)-T_{L, j-2}(x), \quad j=2,3, \ldots, n . \tag{15}
\end{align*}
$$

Definition 1. Let $T_{L, j}(x)$ imply the shifted Chebyshev polynomial of the order $j$ then $T_{L, j}(x)$ can be formulated as [22] and [25].

$$
\begin{equation*}
T_{L, j}(x)=j \sum_{k=0}^{j}(-1)^{i-k} \frac{(j+k-1)!2^{2 k}}{(j-k)!(2 k)!L^{k}} x^{k}, \quad j=1,2,3, \ldots, n, \tag{16}
\end{equation*}
$$

where $T_{L, j}(0)=(-1)^{j}$ and $T_{L, j}(L)=1$. The orthogonality condition is:

$$
\begin{equation*}
\int_{0}^{\mathrm{T}} \mathrm{~T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x}) \mathrm{T}_{\mathrm{L}, \mathrm{k}}(\mathrm{x}) \mathrm{w}_{\mathrm{L}}(\mathrm{x}) \mathrm{dx}=\mathrm{h}_{\mathrm{j}}, \tag{17}
\end{equation*}
$$

where the weight function

$$
\begin{equation*}
\mathrm{w}_{\mathrm{L}}(\mathrm{x})=\frac{1}{\sqrt{\mathrm{Lx}-\mathrm{x}^{2}}} \tag{18}
\end{equation*}
$$

and

$$
h_{j}=\left\{\begin{array}{ll}
\frac{\varepsilon_{j}}{20} \pi & , \mathrm{k}=\mathrm{j},  \tag{19}\\
2, \mathrm{k} \neq \mathrm{j},
\end{array} \quad \varepsilon_{0}=2, \varepsilon_{\mathrm{j}}=1 ; \mathrm{j} \geq 1\right.
$$

Definition 2. Let $v(x, t)$ be function defined for $0<x<L, 0<t \leq \tau$ and then expanded in the terms of the shifted Chebyshev polynomial as [22] and [24]:

$$
\begin{equation*}
\mathrm{v}(\mathrm{x}, \mathrm{t})=\sum_{\mathrm{i}=0}^{\infty} \sum_{\mathrm{j}=0}^{\infty} \mathrm{a}_{\mathrm{ij}} \mathrm{~T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x}) . \tag{20}
\end{equation*}
$$

If the infinite series in Eq. (20) is truncated, then the function $v(x, t)$ can be approximated as:

$$
\begin{equation*}
\mathrm{v}_{\mathrm{m}, \mathrm{n}}(\mathrm{x}, \mathrm{t}) \simeq \sum_{\mathrm{i}=0}^{\mathrm{m}} \sum_{\mathrm{j}=0}^{\mathrm{n}} \mathrm{a}_{\mathrm{ij}} \mathrm{~T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x})=\psi^{\mathrm{T}}(\mathrm{t}) \mathrm{Af}(\mathrm{x}) \tag{21}
\end{equation*}
$$

where the shifted Chebyshev vectors $\psi(t)$ and $\mathfrak{f}(x)$ and the matrix $A$ are given as:

$$
\begin{align*}
& \psi(\mathrm{t})=\left[\mathrm{T}_{\tau, 0}(\mathrm{t}), \mathrm{T}_{\tau, 1}(\mathrm{t}), \ldots, \mathrm{T}_{\tau, \mathrm{m}}(\mathrm{t})\right]^{\mathrm{T}}, \\
& \boldsymbol{f}(\mathrm{x})=\left[\mathrm{T}_{\mathrm{L}, 0}(\mathrm{x}), \mathrm{T}_{\mathrm{L}, 1}(\mathrm{x}), \ldots, \mathrm{T}_{\mathrm{L}, \mathrm{n}}(\mathrm{x})\right]^{\mathrm{T}}, \\
& \mathrm{~A}=\left[\begin{array}{cccc}
\mathrm{a}_{00} & \mathrm{a}_{01} & \cdots & \mathrm{a}_{0 \mathrm{n}} \\
\mathrm{a}_{10} & \mathrm{a}_{11} & \cdots & \mathrm{a}_{1 \mathrm{n}} \\
\vdots & \vdots & \cdots & \vdots \\
\mathrm{a}_{\mathrm{m} 0} & \mathrm{a}_{\mathrm{m} 1} & \cdots & \mathrm{a}_{\mathrm{mn}}
\end{array}\right] . \tag{22}
\end{align*}
$$

Here, the shifted Chebyshev coefficient matrix $A=\left(a_{i j}\right)$ is given by

$$
\begin{equation*}
\mathrm{a}_{\mathrm{ij}}=\frac{1}{\mathrm{~h}_{\mathrm{i}} \mathrm{~h}_{\mathrm{j}}} \int_{0}^{\tau} \int_{0}^{\mathrm{L}} \mathrm{u}(\mathrm{x}, \mathrm{t}) \mathrm{T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x}) \mathrm{w}_{\tau}(\mathrm{t}) \mathrm{w}_{\mathrm{L}}(\mathrm{x}) \mathrm{dxdt}, \quad \mathrm{i}=0,1, \ldots, \mathrm{~m}, \mathrm{j}=0,1, \ldots, \mathrm{n} . \tag{23}
\end{equation*}
$$

We approximate functions $v(x, t), q(x, t)$ and $f(x)$ by using the shifted Chebyshev operational matrix follow as:

$$
\left\{\begin{array}{l}
v_{m, n}(x, t)=\psi^{T}(t) A f(x),  \tag{24}\\
\mathrm{q}_{\mathrm{m}, \mathrm{n}}(\mathrm{x}, \mathrm{t}) \simeq \sum_{\mathrm{i}=0}^{\mathrm{m}} \sum_{\mathrm{j}=0}^{\mathrm{n}} \mathrm{q}_{\mathrm{ij}} \mathrm{~T}_{\mathrm{r}, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x})=\psi^{\mathrm{T}}(\mathrm{t}) \operatorname{Cf}(\mathrm{x}), \\
\mathrm{f}(\mathrm{x}) \simeq \sum_{\mathrm{j}=0}^{\mathrm{n}} \mathrm{f}_{\mathrm{j}}^{\mathrm{L}, \mathrm{j}} \mathrm{f}(\mathrm{x})=\psi^{\mathrm{T}}(\mathrm{t}) \mathrm{Ff}(\mathrm{x}),
\end{array}\right.
$$

where $A$ is an unknown $(m+1) \times(n+1)$ matrix, $Q$ and $F$ are known $(m+1) \times(n+1)$ matrices as:

$$
\mathrm{Q}=\left[\begin{array}{cccc}
\mathrm{q}_{00} & \mathrm{q}_{01} & \cdots & \mathrm{q}_{0 n}  \tag{25}\\
\mathrm{q}_{01} & \mathrm{q}_{11} & \cdots & \mathrm{q}_{1 \mathrm{n}} \\
\vdots & \vdots & \vdots & \vdots \\
\mathrm{q}_{\mathrm{m} 0} & \mathrm{q}_{\mathrm{m} 1} & \cdots & \mathrm{q}_{\mathrm{mn}}
\end{array}\right], \quad \mathrm{F}=\left[\begin{array}{ccccc}
\mathrm{f}_{0} & \mathrm{f}_{1} & \cdots & \mathrm{f}_{\mathrm{n}-1} & f_{\mathrm{n}} \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{array}\right],
$$

where

$$
\begin{equation*}
\mathrm{q}_{\mathrm{ij}}=\frac{1}{\mathrm{~h}_{\mathrm{i}} \mathrm{~h}_{\mathrm{j}}} \int_{0}^{\tau} \int_{0}^{\mathrm{L}} \mathrm{q}(\mathrm{x}, \mathrm{t}) \mathrm{T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x}) \mathrm{w}_{\tau}(\mathrm{t}) \mathrm{w}_{\mathrm{L}}(\mathrm{x}) \mathrm{dxdt}, \quad \mathrm{i}=0,1, \ldots, \mathrm{~m}, \mathrm{j}=0,1, \ldots, \mathrm{n}, \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{f}_{\mathrm{j}}=\frac{1}{h_{\mathrm{j}}} \int_{0}^{\mathrm{L}} \mathrm{f}(\mathrm{x}) \mathrm{T}_{\mathrm{L}, \mathrm{j}}(\mathrm{x}) \mathrm{w}_{\mathrm{L}}(\mathrm{x}) \mathrm{dx}, \quad \mathrm{j}=0,1, \ldots, \mathrm{n} . \tag{27}
\end{equation*}
$$

## 2.2 | Operational Matrices of Derivative and Integral

In this section, Shifted Chebyshev Vectors are used and so as its operational matrices of derivative and integral to solve inverse heat problem of the form Eqs. (8) to (12).

Theorem 1. The derivative of the shifted Chebyshev vector $\mathfrak{f}(x)$ may be expressed by [1], [2], [4], [24].

$$
\begin{equation*}
\frac{\mathrm{d} \mathfrak{f}(\mathrm{x})}{\mathrm{dx}}=\mathrm{D}^{(1)} \mathfrak{f}(\mathrm{x}), \tag{28}
\end{equation*}
$$

where $D^{(1)}=d_{i j}$ is the $(n+1) \times(n+1)$ operational matrix of derivative and given by

$$
D^{(1)}=d_{i j}=\left\{\begin{array}{rr}
\frac{4 i}{\varepsilon_{j} L}, & j=i-k, \begin{cases}k=1,3, \ldots, n, & i f(n) \text { isodd }, \\
k=1,3, \ldots, n-1, & \text { if }(n) \text { iseven }, \\
0, & \text { otherwise. }\end{cases} \tag{29}
\end{array}\right.
$$

where $\varepsilon_{0}=2, \varepsilon_{j}=1, j \geq 1$, see [4] and [24].
Corollary 1. Using Eq. (28), the operational matrix for the nth derivative can be stated as [2], [3], [25].

$$
\begin{equation*}
\frac{\mathrm{d}^{\mathrm{n}} \phi(\mathrm{x})}{\mathrm{dx} \mathrm{x}^{\mathrm{n}}}=\left(\mathrm{D}^{(1)}\right)^{\mathrm{n}} \phi(\mathrm{x}), \tag{30}
\end{equation*}
$$

where $n \in N$ is the nth power of matrix $D^{(1)}$. So we have

$$
\begin{equation*}
D^{n}=\left(D^{(1)}\right)^{n}, n=1,2, \ldots \tag{31}
\end{equation*}
$$

Theorem 2. The integration of $\psi_{\tau, m}(t)$ may be written as [2], [3], [25].

$$
\begin{equation*}
\int_{0}^{\mathrm{t}} \psi\left(\mathrm{t}^{\prime}\right) \mathrm{dt}^{\prime} \simeq \mathrm{P} \psi(\mathrm{t}) \tag{32}
\end{equation*}
$$

where $P$ is the $(m+1) \times(m+1)$ shifted Chebyshev operational matrix of integration and is given by

$$
\mathrm{p}=\left[\begin{array}{cccccccc}
\mathrm{w}_{0} & \delta_{0} & 0 & 0 & 0 & \cdots & 0 & 0  \tag{33}\\
\mathrm{w}_{1} & 0 & \lambda_{1} & 0 & 0 & \cdots & 0 & 0 \\
\mathrm{w}_{2} & \delta_{2} & 0 & \lambda_{2} & 0 & \cdots & 0 & 0 \\
\mathrm{w}_{3} & 0 & \delta_{3} & 0 & \lambda_{3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\mathrm{w}_{\mathrm{m}-2} & 0 & 0 & 0 & \ddots & \ddots & \lambda_{\mathrm{m}-2} & 0 \\
\mathrm{w}_{\mathrm{m}-1} & 0 & 0 & 0 & 0 & \ddots & 0 & \lambda_{\mathrm{m}-1} \\
\mathrm{w}_{\mathrm{m}} & 0 & 0 & 0 & 0 & \cdots & \delta_{\mathrm{m}} & 0
\end{array}\right],
$$

where $w_{k}, \delta_{k}$ and $\lambda_{k}$ obtained using the following formula:

$$
\begin{align*}
& \mathrm{w}_{\mathrm{k}}= \begin{cases}\frac{\tau}{2}, & \mathrm{k}=0, \\
\frac{-\tau}{8}, & \mathrm{k}=1, \\
\frac{(-1)^{k+1} \tau}{2(\mathrm{k}-1)(\mathrm{k}+1)}, & \mathrm{k}=2,3, \ldots\end{cases}  \tag{34}\\
& \lambda_{\mathrm{k}}= \begin{cases}\frac{\tau}{\frac{\tau}{2},} \\
0, & \mathrm{k}=0, \\
\frac{-\tau}{4(\mathrm{k}-1)}, & \mathrm{k}=2,3, \ldots\end{cases} \\
& \frac{\mathrm{l}}{\mathrm{~g}, \tau} \begin{array}{ll}
\frac{\tau}{4(\mathrm{k}+1)}, & \mathrm{k}=2,3, \ldots
\end{array}
\end{align*}
$$

Obviously similar to Eq. (32) we have

$$
\begin{equation*}
\int_{0}^{x} f\left(x^{\prime}\right) d x^{\prime} \simeq G f(x) \tag{35}
\end{equation*}
$$

where $G$ is the $(n+1) \times(n+1)$ shifted Chebyshev operational matrix of integration and is defined similar to Eq. (33).

## 3 | Shifted Chebyshev Tau Method

In this part, SCT method is applicable to solve the inverse problem for heat Eqs. (8) to (12).

Integrating Eq. (8) from 0 to $t$ and using Eq. (9), we have

$$
\begin{equation*}
\mathrm{v}(\mathrm{x}, \mathrm{t})-\mathrm{f}(\mathrm{x})=\int_{0}^{\mathrm{t}} \mathrm{v}_{\mathrm{xx}}\left(\mathrm{x}, \mathrm{t}^{\prime}\right) \mathrm{dt} \mathrm{t}^{\prime}+\int_{0}^{\mathrm{t}} \mathrm{r}\left(\mathrm{t}^{\prime}\right) \mathrm{q}\left(\mathrm{x}, \mathrm{t}^{\prime}\right) \mathrm{dt}^{\prime} \tag{36}
\end{equation*}
$$

Using Eq. (24), Corollary 1 and Theorem 2 we obtain

$$
\begin{equation*}
\int_{0}^{t} v_{x x}\left(x, t^{\prime}\right) d t^{\prime}=\left(\int_{0}^{t} \psi^{T}\left(t^{\prime}\right) d t^{\prime}\right) A\left(\frac{d^{2} f(x)}{d x^{2}}\right)=\psi^{T}(t) P^{T} A D^{2} f(x) . \tag{37}
\end{equation*}
$$

The function $r(t)$ may be expanded in terms of $m+1$ shifted Chebyshev series as (see [3] and [24]).

$$
\begin{equation*}
\mathrm{r}(\mathrm{t}) \simeq \mathrm{r}_{\mathrm{m}}(\mathrm{t})=\sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{~b}_{\mathrm{k}} \mathrm{~T}_{\tau, \mathrm{k}}(\mathrm{t})=\mathrm{B}^{\mathrm{T}} \psi(\mathrm{t}), \tag{38}
\end{equation*}
$$

where $B=\left[b_{0}, b_{1}, \ldots, b_{m}\right]^{T}$ is an unknown vector.
Now, using Eqs. (19), (30) and (38) we have

$$
\begin{equation*}
\int_{0}^{t} r\left(\mathrm{t}^{\prime}\right) q\left(x, \mathrm{t}^{\prime}\right) \mathrm{dt}^{\prime}=\left(\int_{0}^{\mathrm{t}} \mathrm{~B}^{\mathrm{T}} \psi\left(\mathrm{t}^{\prime}\right) \psi^{\mathrm{T}}\left(\mathrm{t}^{\prime}\right) \mathrm{dt}^{\prime}\right) \mathrm{Qf}(\mathrm{x}) . \tag{39}
\end{equation*}
$$

Let us set

$$
\begin{equation*}
\mathrm{B}^{\mathrm{T}} \psi(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t})=\psi^{\mathrm{T}}(\mathrm{t}) \mathrm{H}, \tag{40}
\end{equation*}
$$

where $H$ is an $(m+1) \times(m+1)$ matrix. To find $H$, we rewrite Eq. (40) (see [2] and [3]) in the form

$$
\begin{equation*}
\sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{~b}_{\mathrm{k}} \mathrm{~T}_{\tau, \mathrm{k}}(\mathrm{t}) \mathrm{T}_{\tau, j}(\mathrm{t})=\sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{H}_{\mathrm{kj}} \mathrm{~T}_{\tau, \mathrm{k}}(\mathrm{t}), \quad \mathrm{j}=0,1, \ldots, \mathrm{~m} . \tag{41}
\end{equation*}
$$

Multiplying both sides of $E q$. (41) by $T_{\tau, i}(t) w_{\tau}(t), i=0,1, \ldots, m$ and integrating from 0 to $\tau$ yields

$$
\begin{align*}
& \sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{~b}_{\mathrm{k}} \int_{0}^{\tau} \mathrm{T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\tau, \mathrm{k}}(\mathrm{t}) \mathrm{T}_{\tau, \mathrm{j}}(\mathrm{t}) \mathrm{w}_{\tau}(\mathrm{t}) \mathrm{dt}=\sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{H}_{\mathrm{k}} \int_{0}^{\tau} \mathrm{T}_{\tau, \mathrm{k}}(\mathrm{t}) \mathrm{T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{w}_{\tau}(\mathrm{t}) \mathrm{dt} .  \tag{42}\\
& \hline, \ldots, \mathrm{m} .
\end{align*}
$$

Byusing Eq. (42) and employing the orthogonality relation Eq. (8) gives:

$$
\sum_{\mathrm{k}=0}^{m} \mathrm{~b}_{\mathrm{k}} \int_{0}^{\tau} \mathrm{T}_{\tau, \mathrm{i}}(\mathrm{t}) \mathrm{T}_{\tau, \mathrm{k}}(\mathrm{t}) \mathrm{T}_{\tau, \mathrm{j}}(\mathrm{t}) \mathrm{w}_{\tau}(\mathrm{t}) \mathrm{dt}=\mathrm{H}_{\mathrm{ij}} \mathrm{~h}_{\mathrm{i}},
$$

or equivalently

$$
\begin{equation*}
H_{i j}=\frac{1}{h_{i}} \sum_{k=0}^{m} b_{k} \int_{0}^{\tau} T_{\tau, i}(t) T_{\tau, k}(t) T_{\tau, j}(t) w_{\tau}(t) d t, \quad i, j=0,1, \ldots, m . \tag{43}
\end{equation*}
$$

Employing Eqs. (32), (39) and (40) can be written as:

$$
\begin{equation*}
\int_{0}^{t} r\left(t^{\prime}\right) q\left(x, t^{\prime}\right) d t^{\prime}=\psi^{T}(t) P^{T} H Q f(x) . \tag{44}
\end{equation*}
$$

Applying Eqs. (21), (24), (37) and (44) the residual $\operatorname{Res}_{m, n}(x, t)$ for Eq. (36) can be written as:

$$
\operatorname{Res}_{m, n}(x, t)=\psi^{T}(t)\left[A-F-P^{T} H Q-P^{T} A D^{2}\right] f(x) .
$$

Employing standard tau method, generate $(m+1) \times(n-1)$ linear algebraic equations using the following algebraic equations:

$$
\begin{equation*}
\int_{0}^{\tau} \int_{0}^{\mathrm{L}} \operatorname{Res}_{m, n}(x, t) T_{\tau, i}(t) T_{L, j}(x) d x d t=0, \quad i=0,1, \ldots, m, j=0,1, \ldots, n-2 . \tag{45}
\end{equation*}
$$

Also, by substituting Eqs. (24) and (38) in Eqs. (10) and (11) we get

$$
\begin{align*}
& \alpha_{1}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{ADf}(0)+\beta_{1}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{Af}(0)+\gamma_{1}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{Af}(\mathrm{L})=\mathrm{g}_{1}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \mathrm{B}, \\
& \alpha_{2}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{ADf}(\mathrm{L})+\beta_{2}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{Af}(0)+\gamma_{2}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \operatorname{Af}(\mathrm{L})=\mathrm{g}_{2}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) B . \tag{46}
\end{align*}
$$

And applying Eqs. (24) and (35) in Eq. (12) we have

$$
\begin{equation*}
\psi^{\mathrm{T}}(\mathrm{t}) \operatorname{AGf}(\mathrm{s}(\mathrm{t}))=\mathrm{E}(\mathrm{t}) \psi^{\mathrm{T}}(\mathrm{t}) \mathrm{B} . \tag{47}
\end{equation*}
$$

Eqs. (46) to (47) are collocated at $m+1$ points. Here, the roots of $T_{\tau, \mathrm{m}+1}(t)$ are used as a collocation points. Eqs. (45) to (47) yields a set of $(m+1)(n+1)+(m+1)$ algebraic equations which can be solved for $a_{i j}, i=0,1, \ldots, m, j=0,1, \ldots, n$ and $b_{k}, k=0,1, \ldots, m$. Consequently $v(x, t)$ given in $E q$. (21) and $r(t)$ given in Eq. (38) can be calculated. Finally using Eqs. (13) and (14), $u(x, t)$ and $p(t)$ can be found.

## 4 | Error Bound

In this section, an upper bound of the absolute errors will be given by using Lagrange interpolation polynomials. Our aim is to obtain an analytic expression for the error of the best approximation of a smooth function $v(x, t)$ and source function $r(t)$ by them expansion in terms of shifted Chebyshev polynomials.

Theorem 3. If $v(x, t) \in \Omega \equiv[0, L] \times[0, \tau]$ is a sufficiently smooth function and $v_{m, n}(x, t)$ is the interpolating polynomial for $v(x, t)$ at points $\left(x_{j}, t_{i}\right)$ where $x_{j}, O \leq j \leq n$ are the roots of the $T_{L, j}(x)$ in $[0, L]$ and $t_{i}, 0 \leq i \leq m$ are the roots of the $T_{\tau, i}(t)$ in $[0, \tau]$, then the error bound is presented as follows:

$$
\left|v(x, t)-v_{m, n}(x, t)\right| \leq K_{1} \frac{\left(\frac{L}{2}\right)^{n+1}}{(n+1)!2^{n}}+K_{2} \frac{\left(\frac{\tau}{2}\right)^{m+1}}{(m+1)!2^{m}}+K_{3} \frac{\left(\frac{L}{2}\right)^{n+1}\left(\frac{\tau}{2}\right)^{m+1}}{(n+1)!(m+1)!2^{m+n}}
$$

Proof: Let us define the error function $v(x, t)-v_{m, n}(x, t)$, then by similar procedures as in [26], we have

$$
\begin{align*}
& v(x, t)-v_{m, n}(x, t)=\frac{\partial^{n+1} v(\xi, t)}{\partial x^{n+1}(n+1)!} \prod_{j=0}^{n}\left(x-x_{j}\right)+\frac{\partial^{m+1} v(x, \eta)}{\partial t^{m+1}(m+1)!} \prod_{i=0}^{m}\left(t-t_{i}\right)- \\
& \frac{\partial^{m+n+2} v\left(\xi^{\prime}, \eta^{\prime}\right)}{\partial x^{n+1} \partial t^{m+1}(m+1)!(n+1)!} \prod_{j=0}^{m}\left(x-x_{j}\right) \prod_{i=0}^{m}\left(t-t_{i}\right), \tag{48}
\end{align*}
$$

where $\xi, \xi^{\prime} \in[0, L]$ and $\eta, \eta^{\prime} \in[0, \tau]$. Therefore

$$
\begin{align*}
& \left|v(x, t)-v_{m, n}(x, t)\right|=\max _{(x, t) \in \Omega}\left|\frac{\partial^{n+1} v(x, t)}{\sum_{m} \partial x^{n+1}}\right| \frac{\prod_{j=0}^{n}\left|x-x_{j}\right|}{(n+1)!}+\max _{(x, t) \in \Omega} \frac{\left|\partial^{m+1} v(x, t)\right|}{\prod_{i=0}^{m}\left|t-t_{i}\right|}  \tag{49}\\
& \max _{(x, t) \in \Omega}^{m+1} \\
& (m+1)! \\
& \left|\frac{\partial^{m+n+2} v(x, t)}{\partial x^{m+1} \partial t^{n+1}}\right| \frac{\prod_{i=0}^{m}\left|x-x_{j}\right| \prod_{i=0}\left|t-t_{i}\right|}{(m+1)!(n+1)!} .
\end{align*}
$$

Assume that there are constants $K_{1}, K_{2}$ and $K_{3}$, such that

$$
\begin{align*}
& \max _{(x, t) \in \Omega}\left|\frac{\partial^{n+1} v(x, t)}{\partial x^{n+1}}\right| \leq K_{1}, \\
& \max _{(x, t) \in \Omega}\left|\frac{\partial^{m+1} v(x, t)}{\partial t^{m+1}}\right| \leq K_{2},  \tag{50}\\
& \max _{(x, t) \in \Omega}\left|\frac{\partial^{n+m+2} v(x, t)}{\partial x^{n+1} \partial t^{m+1}}\right| \leq K_{3} .
\end{align*}
$$

Let us use the one-to-one mapping $x=2 L(z+1)$ between the intervals $[-1,1]$ and $[0, L]$ to deduce and also taking into account the estimates for Chebyshev interpolation nodes, then we obtain:

$$
\begin{align*}
& \min _{x_{i} \in\{0, L] \mid 0 \times x \leq L} \max _{\substack{0 \\
n+1}}\left|\prod_{j=0}^{n}\left(x-x_{j}\right)\right|=\min _{z_{i} \in[-1,1]-1 \leq \leq \leq 1} \max _{1}\left|\prod_{j=q_{h}}^{n} \frac{L}{2}\left(z-z_{j}\right)\right| \\
& =\left(\frac{L}{2}\right)^{n+1} \min _{z_{i}[-1,1,]-1 \leq z \leq 1}^{n-1}\left|\prod_{j=0}^{n}\left(z-z_{j}\right)\right|=\left(\frac{L^{2}}{2}\right)^{i=a_{h}+1} \frac{1}{2^{n}} \text {. } \tag{51}
\end{align*}
$$

Now, by replacing Eqs. (50) and (51) in Eq. (49), yields the following desired result:

$$
\begin{align*}
& \max _{(x, t) \in \Omega}\left|\frac{\partial^{n+1} v(x, t)}{\partial x^{n+1}}\right| \leq K_{1}, \\
& \max _{(x, t) \in \Omega}\left|\frac{\partial^{n+m+2} v(x, t)}{\partial x^{n+1} \partial t^{m+1}}\right| \leq K_{3}, \tag{52}
\end{align*}
$$

or equivalently

$$
\begin{equation*}
\left|v(x, t)-v_{m, n}(x, t)\right| \leq K_{1} \frac{L^{n+1}}{(n+1)!2^{2 n+1}}+K_{2} \frac{\tau^{m+1}}{(m+1)!2^{2 m+1}}+K_{3} \frac{L^{n+1} \tau^{m+1}}{(n+1)!(m+1)!2^{2 m+2 n+1}} . \tag{53}
\end{equation*}
$$

Therefore, an upper bound of the absolute errors is obtained for the approximate and exact solutions.

Remark 1. In the special case if $\boldsymbol{m}=\boldsymbol{n}$ and $L=\tau=1$ we have

$$
\begin{equation*}
\left|v(x, t)-v_{n, n}(x, t)\right| \leq\left(K_{1}+K_{2}+\frac{K_{3}}{(n+1)!2^{2 n+1}}\right) \frac{1}{(n+1)!2^{2 n+1}} . \tag{54}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathrm{K}_{1}+\mathrm{K}_{2}+\frac{\mathrm{K}_{3}}{(\mathrm{n}+1)!2^{2 \mathrm{n}+1}}=\alpha . \tag{55}
\end{equation*}
$$

So that we may write

$$
\begin{equation*}
\left|v(x, t)-v_{n, n}(x, t)\right| \leq \frac{\alpha}{(n+1)!2^{2 n+1}} \tag{56}
\end{equation*}
$$

Hence, show that

$$
\begin{equation*}
\left|v(x, t)-v_{n, n}(x, t)\right|=O\left(\frac{1}{(n+1)!2^{2 n+1}}\right) \tag{57}
\end{equation*}
$$

Theorem 4. Let $r_{m}(t)=\sum_{k=0}^{m} b_{k} T_{\tau, k}(t)$, be the shifted chebyshev functions expansion of the real sufficiently smooth function $r(t)$ and there is a real number $K$ such that

$$
\begin{equation*}
\left\|\mathrm{r}(\mathrm{t})-\mathrm{r}_{\mathrm{m}}(\mathrm{t})\right\|_{2} \leq \mathrm{K} \frac{\tau^{\mathrm{m}+1}}{(\mathrm{~m}+1)!2^{2 \mathrm{~m}+1}} \tag{58}
\end{equation*}
$$

Moreover, if $\bar{r}_{m}(t)=\sum_{k=0}^{m} \bar{b}_{k} T_{\tau, k}(t)$ be an approximation for the shifted chebyshev functions $r_{m}(t)$, then there are $\mu_{\tau}$ and $\lambda_{\tau}$ such that

$$
\begin{equation*}
\left\|\mathrm{r}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})\right\|_{2} \leq \mu_{\tau}\left(\frac{\mathrm{K} \tau^{\mathrm{m}+1}}{2^{2 \mathrm{~m}+1}(\mathrm{~m}+1)!}\right)+\lambda_{\tau}\|\mathrm{B}-\overline{\mathrm{B}}\|_{2} . \tag{59}
\end{equation*}
$$

Proof: To prove Eq. (59), we write

$$
\begin{equation*}
\mathrm{r}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})=\mathrm{r}(\mathrm{t})-\mathrm{r}_{\mathrm{m}}(\mathrm{t})+\mathrm{r}_{\mathrm{m}}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t}) \tag{60}
\end{equation*}
$$

Satisfies the triangle inequality

$$
\begin{equation*}
\left\|\mathrm{r}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})\right\|_{2} \leq\left\|\mathrm{r}(\mathrm{t})-\mathrm{r}_{\mathrm{m}}(\mathrm{t})\right\|_{2}+\left\|\mathrm{r}_{\mathrm{m}}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})\right\|_{2}, \tag{61}
\end{equation*}
$$

The right-hand inequality in Eq. (61) write as follow:

$$
\begin{align*}
& \left\|r(t)-r_{m}(t)\right\|_{2}=\left(\int_{0}^{\tau} w_{\tau}(t)\left|r(t)-r_{m}(t)\right|^{2} d t\right)^{\frac{1}{2}} \leq\left(\int_{0}^{\tau} w_{\tau}(t)\left(\frac{K \tau^{m+1}}{2^{2 m+1}(m+1)!}\right)^{2} d t\right)^{\frac{1}{2}}  \tag{62}\\
& =\left(\frac{K \tau^{m+1}}{2^{2 m+1}(m+1)!}\right)\left(\int_{0}^{\tau} w_{\tau}(t) d t\right)^{\frac{1}{2}}=\mu_{\tau}\left(\frac{K \tau^{m+1}}{2^{2 m+1}(m+1)!}\right)
\end{align*}
$$

where

$$
\mu_{\tau}=\frac{1}{4} \sqrt{t-t^{2}}(2 t-1)-\sin ^{-1}(\sqrt{1-t}) .
$$

Moreover

$$
\begin{align*}
& \left\|\mathrm{r}_{\mathrm{m}}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})\right\|_{2}=\left(\int_{0}^{\tau} \mathrm{w}_{\tau}(\mathrm{t})\left[\sum_{\mathrm{k}=0}^{\mathrm{m}}\left(\mathrm{~b}_{\mathrm{k}}-\overline{\mathrm{b}}_{\mathrm{k}}\right) \mathrm{T}_{\tau, k}(\mathrm{t})\right]^{2} \mathrm{dt}\right)^{\frac{1}{2}} \\
& \leq\left(\int_{0}^{\tau} \mathrm{w}_{\tau}(\mathrm{t})\left[\sum_{\mathrm{k}=0}^{m}\left|\mathrm{c}_{\mathrm{i}}-\overline{\mathrm{c}}_{\mathrm{i}}\right|^{2}\right]\left[\sum_{\mathrm{k}=0}^{m}\left|T_{\tau, k}(\mathrm{t})\right|^{2}\right] \mathrm{dt}\right)^{\frac{1}{2}}  \tag{63}\\
& =\left(\sum_{\mathrm{k}=0}^{\mathrm{m}}\left|\mathrm{~b}_{\mathrm{k}}-\overline{\mathrm{b}}_{\mathrm{k}}\right|^{2}\right)^{\frac{1}{2}}\left(\int_{0}^{\tau} \mathrm{w}_{\tau}(\mathrm{t})\left[\sum_{\mathrm{k}=0}^{m}\left|\mathrm{~T}_{\tau, k}(\mathrm{t})\right|^{2}\right] \mathrm{dt}\right)^{\frac{1}{2}}=\|\mathrm{B}-\overline{\mathrm{B}}\|_{2}\left(\sum_{\mathrm{k}=0}^{\mathrm{m}} \mathrm{~h}_{\tau, k}\right)^{\frac{1}{2}}=\lambda_{\tau}\|\mathrm{B}-\overline{\mathrm{B}}\|_{2},
\end{align*}
$$

where

$$
\lambda_{\tau}=\left(\sum_{k=0}^{m} h_{\tau, k}\right)^{\frac{1}{2}} .
$$

By summing relation Eqs. (62) and (63) upper-bound in Theorem 4 the following relation is created:

$$
\left\|\mathrm{r}(\mathrm{t})-\overline{\mathrm{r}}_{\mathrm{m}}(\mathrm{t})\right\|_{2} \leq \mu_{\tau}\left(\frac{\mathrm{K} \tau^{\mathrm{m}+1}}{2^{2 \mathrm{~m}+1}(\mathrm{~m}+1)!}\right)+\lambda_{\tau}\|\mathrm{B}-\overline{\mathrm{B}}\|_{2} .
$$

This completes the proof.

## 5 | Numerical Results

In this section, numerical experiment is chosen to illustrate the efficiency and performance of the SCT method for solving Eq. (1) with Conditions (1) to (4) and energy Condition (5). In this case, the exact solution of the problem is known. The accuracy of our approach is estimated by the following error functions:

$$
e_{u}=\left|u_{m, n}(x, t)-u(x, t)\right|, \quad e_{p}=\left|p_{m}(\mathrm{t})-\mathrm{p}(\mathrm{t})\right| .
$$

Example 1. Considers (1) to (5) with the given data:

$$
\begin{aligned}
& \tau=0.5, L=1, \\
& q(x, t)=\left(1-t^{3}\right) \sin x-x^{2}(t-1)^{2} \exp \left(t^{2}\right)-2 \exp \left(t^{2}\right)-t^{2}\left(\pi \cos x+t^{3}+t-3\right), \\
& f(x)=x^{2}+\pi \cos x, \\
& g_{1}(t)=\pi+t^{3}, \\
& g_{2}(t)=t \sin 1+\exp \left(t^{2}\right)+\pi \cos 1+t^{3}, \\
& \alpha_{1}=0, \alpha_{2}=0, \\
& \beta_{1}(t)=1, \beta_{2}(t)=0, \\
& \gamma_{1}(t)=0, \gamma_{2}(t)=1, \\
& E(t)=(\pi \sin t-t \cos t) \cos (\sin t)+(t \sin t+\pi \cos t) \sin (\sin t)+ \\
& \frac{1}{3} \exp \left(t^{2}\right)\left(\sin { }^{3} t+t^{3}\right)+\left(t^{2} \sin t+\operatorname{tsin}^{2} t\right) \exp \left(t^{2}\right)+\left(1+t^{3}+t^{2} \sin t\right) t, \\
& s(t)=t+\sin t .
\end{aligned}
$$

For which the exact solution is [27]

$$
u(x, t)=t \sin x+x^{2} \exp \left(t^{2}\right)+\pi \cos x+t^{3}
$$

and

$$
\mathrm{p}(\mathrm{t})=1+\mathrm{t}^{2}
$$

In Table 1, we display error function $\left|u_{m, n}(x, t)-u(x, t)\right|$, using the proposed method at $\mathrm{t}=0.25$ with m $=\mathrm{n}=4,6,8$. Also, the results obtained for $\left|p_{\mathrm{m}}(t)-p(t)\right|$ are listed in Table 2. In Fig. 1, the space-time graph of exact solution $u(x, t)$ and time graph of $p(t)$ are plotted. In Fig. 2 and Fig. 3 graph of the absolute error $u(x, 0.25)$ for $x=0.1$ and $x=0.9$ with various value of $m=n$ and $\left|p_{m}(t)-p(t)\right|$ for $t=0.1$ and $\mathrm{t}=0.5$ with various value of m are shown respectively. Fig. 4 and Fig. 5 Graph of the absolute error for $u(x, 0.25)$ with $m=n=4,6,8$ and $\left|p_{m}(t)-p(t)\right|$ for $m=n=4,6,8$ obtained.

Table 1. Error function $\left|\mathrm{u}_{\mathrm{m}, \mathrm{n}}(\mathrm{x}, \mathrm{t})-\mathrm{u}(\mathrm{x}, \mathrm{t})\right|$ with $\mathrm{t}=0.25$.

| x | $\mathrm{m}=\mathrm{n}=4$ | Error <br> $\mathrm{m}=\mathrm{n}=6$ | $\mathrm{~m}=\mathrm{n}=8$ |
| :--- | :--- | :--- | :--- |
| 0.1 | $7.33 \times 10^{-5}$ | $1.88 \times 10^{-6}$ | $2.25 \times 10^{-8}$ |
| 0.2 | $4.85 \times 10^{-5}$ | $7.52 \times 10^{-9}$ | $2.50 \times 10^{-10}$ |
| 0.3 | $1.15 \times 10^{-5}$ | $1.44 \times 10^{-7}$ | $5.64 \times 10^{-10}$ |
| 0.4 | $6.99 \times 10^{-5}$ | $1.56 \times 10^{-6}$ | $1.87 \times 10^{-8}$ |
| 0.5 | $5.74 \times 10^{-4}$ | $3.95 \times 10^{-5}$ | $6.68 \times 10^{-6}$ |
| 0.6 | $2.31 \times 10^{-4}$ | $4.01 \times 10^{-6}$ | $3.54 \times 10^{-8}$ |
| 0.7 | $1.89 \times 10^{-4}$ | $1.99 \times 10^{-6}$ | $2.15 \times 10^{-8}$ |
| 0.8 | $5.49 \times 10^{-5}$ | $1.78 \times 10^{-7}$ | $5.47 \times 10^{-9}$ |

Table 2. Error function $\left|p_{m}(t)-p(t)\right|$.

| $\mathbf{t}$ | $\mathrm{m}=\mathrm{n}=4$ | Error <br> $\mathrm{m}=\mathrm{n}=6$ | $\mathrm{~m}=\mathrm{n}=8$ |
| :--- | :---: | :---: | :---: |
| 0.05 | $2.29 \times 10^{-3}$ | $5.76 \times 10^{-5}$ | $6.99 \times 10^{-7}$ |
| 0.1 | $2.60 \times 10^{-4}$ | $4.33 \times 10^{-6}$ | $1.34 \times 10^{-8}$ |
| 0.15 | $3.21 \times 10^{-3}$ | $4.81 \times 10^{-5}$ | $2.57 \times 10^{-7}$ |
| 0.2 | $8.10 \times 10^{-4}$ | $1.81 \times 10^{-5}$ | $2.07 \times 10^{-7}$ |
| 0.25 | $3.37 \times 10^{-3}$ | $2.12 \times 10^{-4}$ | $3.92 \times 10^{-6}$ |
| 0.3 | $1.09 \times 10^{-3}$ | $1.99 \times 10^{-6}$ | $1.67 \times 10^{-8}$ |
| 0.35 | $8.69 \times 10^{-3}$ | $9.08 \times 10^{-5}$ | $9.77 \times 10^{-7}$ |
| 0.4 | $3.04 \times 10^{-3}$ | $9.05 \times 10^{-6}$ | $3.93 \times 10^{-7}$ |
| 0.45 | $4.08 \times 10^{-3}$ | $9.14 \times 10^{-5}$ | $3.01 \times 10^{-5}$ |
| 0.5 | $4.12 \times 10^{-3}$ | $9.84 \times 10^{-5}$ | $3.76 \times 10^{-5}$ |



Fig. 1. The space-time graph of exact solution $u(x, t)$ (left) and time graph of $p(t)$ (right).


Fig. 2. Graph of the absolute error $u(x, 0.025)$ for $x=0.1$ and $x=0.9$ with various value of $m=n$.


Fig. 3. Plot of $\left|\mathrm{p}_{\mathrm{m}}(\mathrm{t})-\mathrm{p}(\mathrm{t})\right|$ for $\mathrm{t}=0.1$ and $\mathrm{t}=0.5$ with various value of m


Fig. 4. Graph of the absolute error for $\mathrm{u}(\mathrm{x}, 0.25)$ with $\mathrm{m}=\mathrm{n}=4,6,8$.


Fig. 5. Plot of $\left|\mathrm{p}_{\mathrm{m}}(\mathrm{t})-\mathrm{p}(\mathrm{t})\right|$ with $\mathrm{m}=4,6,8$.
Example 2. We consider the second inverse Problems (1) to (5) with;

$$
\begin{aligned}
& \tau=1, L=1 \\
& q(x, t)=0, \\
& f(x)=1+\cos x, \\
& g_{1}(t)=t^{2}\left(1+e^{-t} \sin 1\right) \exp \left(t^{2}-\sin t\right), \\
& g_{2}(t)=\left(t\left(1+e^{-t} \cos 1\right)-e^{-t} \sin 1\right) \exp \left(t^{2}-\sin t\right), \\
& \alpha_{1}=1, \alpha_{2}=1, \\
& \beta_{1}(t)=t^{2}, \beta_{2}(t)=0, \\
& \gamma_{1}(t)=0, \gamma_{2}(t)=1, \\
& E(t)=\left(1+e^{-t} \sin 1\right) \exp \left(t^{2}-\sin t\right), \\
& s(t)=1 .
\end{aligned}
$$

For which the exact solution is [17]:

$$
u(x, t)=\left(1+e^{-t} \cos x\right) \exp \left(t^{2}-\sin t\right)
$$

and

$$
\mathrm{p}(\mathrm{t})=2 \mathrm{t}-\cos \mathrm{t} .
$$

In Table 3, we display error function $\left|u_{m, n}(x, t)-u(x, t)\right|$, using the proposed method at $t=0.5$ with $m=$ $n=4,6,8$. Also, the results obtained for $\left|p_{m}(t)-p(t)\right|$ are listed in Table 4. In Fig. 6, the space-time graph of exact solution $u(x, t)$ and time graph of $p(t)$ are plotted. In Fig. 7 and Fig. 8 Graph of the absolute error $u(x, 0.5)$ for $\mathrm{x}=0.1$ and $\mathrm{x}=0.9$ with various value of $\mathrm{m}=\mathrm{n}$ and $\left|p_{\mathrm{m}}(t)-p(t)\right|$ for $\mathrm{t}=0.1$ and $\mathrm{t}=1$ with various value of m are shown respectively. Fig. 9 and Fig. 10 graph of the absolute error for $u(x, 0.5)$ with $\mathrm{m}=\mathrm{n}=4,6,8$ and $\left|p_{\mathrm{m}}(t)-p(t)\right|$ for $\mathrm{m}=\mathrm{n}=4,6,8$ obtained.

Table 3. Error function $\left|\mathrm{u}_{\mathrm{m}, \mathrm{n}}(\mathrm{x}, \mathrm{t})-\mathrm{u}(\mathrm{x}, \mathrm{t})\right|$ with $\mathrm{t}=0.5$.

| x | $\mathrm{m}=\mathrm{n}=4$ | Error <br> $\mathrm{m}=\mathrm{n}=6$ | $\mathrm{~m}=\mathrm{n}=8$ |
| :--- | :--- | :--- | :--- |
| 0.1 | $1.49 \times 10^{-4}$ | $3.82 \times 10^{-6}$ | $4.58 \times 10^{-8}$ |
| 0.2 | $9.83 \times 10^{-5}$ | $1.53 \times 10^{-8}$ | $5.1 \times 10^{-10}$ |
| 0.3 | $7.38 \times 10^{-5}$ | $9.23 \times 10^{-7}$ | $3.69 \times 10^{-9}$ |
| 0.4 | $1.07 \times 10^{-4}$ | $2.38 \times 10^{-6}$ | $2.86 \times 10^{-8}$ |
| 0.5 | $2.22 \times 10^{-4}$ | $1.53 \times 10^{-5}$ | $2.59 \times 10^{-6}$ |
| 0.6 | $2.17 \times 10^{-5}$ | $3.77 \times 10^{-7}$ | $3.32 \times 10^{-9}$ |
| 0.7 | $-8.76 \times 10^{-5}$ | $-9.21 \times 10^{-7}$ | $-9.95 \times 10^{-9}$ |
| 0.8 | $-8.44 \times 10^{-6}$ | $-2.74 \times 10^{-8}$ | $-8.41 \times 10^{-10}$ |
| 0.9 | $-9.25 \times 10^{-5}$ | $-2.22 \times 10^{-6}$ | $-7.77 \times 10^{-8}$ |

Table 4. Error function $\left|p_{m}(t)-p(t)\right|$.

| t | $\mathrm{m}=\mathrm{n}=4$ | Error <br> $\mathrm{m}=\mathrm{n}=6$ | $\mathrm{~m}=\mathrm{n}=8$ |
| :--- | :--- | :--- | :--- |
| 0.1 | $7.48 \times 10^{-4}$ | $1.17 \times 10^{-5}$ | $2.24 \times 10^{-7}$ |
| 0.2 | $3.80 \times 10^{-4}$ | $7.47 \times 10^{-8}$ | $2.49 \times 10^{-9}$ |
| 0.3 | $3.73 \times 10^{-4}$ | $4.48 \times 10^{-6}$ | $1.79 \times 10^{-8}$ |
| 0.4 | $3.39 \times 10^{-4}$ | $4.20 \times 10^{-5}$ | $1.44 \times 10^{-7}$ |
| 0.5 | $1.48 \times 10^{-3}$ | $7.43 \times 10^{-5}$ | $1.26 \times 10^{-8}$ |
| 0.6 | $3.04 \times 10^{-4}$ | $5.80 \times 10^{-6}$ | $1.19 \times 10^{-8}$ |
| 0.7 | $4.27 \times 10^{-4}$ | $3.49 \times 10^{-6}$ | $4.48 \times 10^{-8}$ |
| 0.8 | $4.12 \times 10^{-5}$ | $1.34 \times 10^{-7}$ | $4.11 \times 10^{-9}$ |
| 0.9 | $4.14 \times 10^{-4}$ | $1.49 \times 10^{-5}$ | $5.81 \times 10^{-7}$ |
| 1 | $4.84 \times 10^{-4}$ | $1.38 \times 10^{-5}$ | $4.53 \times 10^{-7}$ |



Fig. 6. The space-time graph of exact solution $u(x, t)$ (left) and time graph of $\mathrm{p}(\mathrm{t})$ (right).

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Fig. 7. Graph of the absolute error $u(x, 0.5)$ for $x=0.1$ and $x=0.9$ with various value of $m=n$


Fig. 8. Plot of $\left|p_{m}(t)-p(t)\right|$ for $t=0.1$ and $t=1$ with various value of $m$.


Fig. 9. Graph of the absolute error for $\mathrm{u}(\mathrm{x}, 0.5)$ with $\mathrm{m}=\mathrm{n}=4,6,8$.


The obtained results from Tables 1 to 4 showed that this approach can solve the problem effectively. The described computational method produces very accurate results even when employing a small number of collocation points. And also, Figs. 2 to 5 and Figs. 7 to 10 show the reduction in the error for the function $\mathrm{u}(\mathrm{x}, \mathrm{t})$ and control parameter $\mathrm{p}(\mathrm{t})$ by increasing the value of $\mathrm{m}, \mathrm{n}$.

## 6 | Conclusion

In this study, the inverse problem for heat equation is discussed. The SCT method is presented to solve the equation. The numerical approach is to expand the unknown function and unknown control parameter in terms of the shifted chebyshev of the first kind and the tau method so that it reduces the problem into a system of algebraic equation. The obtained results showed that this approach can solve the problem effectively. The new described computational technique produces very accurate results even when a small number of collocation points are employed.

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