Journal of Applied Research on Industrial Engineering

Vol. 1, No. 1 (2014) 1-11

Journal of
Applied Research on Industrial Engineering

Determining the inefficient space and ranking of DMUs with undesirable outputs

M. Bashirzadeh[1a], S. Daneshvar[b], N. Azarmir[c]

[a] *Department of Applied Mathematics, Islamic Azad University Tabriz Branch, Iran.*
[b] *Department of Applied Mathematics Islamic Azad University Tabriz Branch, Iran.*
[c] *Department of Applied Mathematics, Islamic Azad University Tabriz Branch, Iran.*

a r t i c l e i n f o

Article History:
Received: 12 February 2014
Received in revised format: 29 March 2014
Accepted: 21 April 2014
Available online: 29 April 2014

Keywords:
Ranking
Undesirable outputs
Facet Analysis
Data Envelopment Analysis (DEA)

a b s t r a c t

One of the applications of Data Envelopment Analysis (DEA) is in ranking of Decision Making Units (DMUs). When some DMUs are the same in efficiency score, this ranking results in failure. Various methods are introduced to rank efficient and inefficient DMUs and attempt to give a fully ranking in order to improve the evaluation. Many articles are published in this field so that have some problems. In this paper, by considering undesirable outputs and extending the inefficient space, a complete ranking of DMUs is presented. On the other hands using facet as a complement of previous methods leads to a fully ranking.

1. Introduction

Data Envelopment analysis (DEA) is a methodology developed by Charnes et al., 1978; for measuring the relative efficiency of Decision Making Units (DMUs) that have multiple inputs and outputs. Their initial model which is commonly referred to as a CCR model, assumed constant returns to scale. Following CCR model, ranking of DMUs is developed according to efficiency score of DMUs. In the

[1] Corresponding Author name: M. Bashirzadeh, *(md_bashirzadeh@yahoo.com)*

following researches the problem of getting same score efficiency through evaluation was discussed.in order to In order to eliminate this problem; many researchers have done a lot of work to achieve a reasonable ranking of DMUs.

In this paper a new scheme are proposed where by bounding of weights in Dong Guo and Jie Wu's model, i.e. with modification of Dong Guo and Jie Wu's model based on facet analysis, useful results are obtained with respect to this model.

The rest of the paper is organized as follow: section 2 briefly reviews ranking models of Alirezaee and Wu and DongGuo. The presented method of this paper is proposed in section3. A numerical example is presented to illustrate the proposed method in section 4. The paper ends with some conclusions.

2. Research background

2.1. Alirezaee and Afsharian's model

Assuming there are n DMUs with m inputs and s outputs, then, the radial CRS (constant return to scale) DEA efficiency score for DMUp can be computed by the following CCR multiplier model:

$$\begin{aligned} \max\ & EFF'_p = \sum_{r=1}^{s} u_r y_{rp} \\ \text{s.t}\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ \ j = 1,\dots,n; \\ & \sum_{i=1}^{m} v_i x_{ip} = 1, \\ & u_r, v_i \ge 0, \quad r = 1,\dots,s;\ i = 1,\dots,m \end{aligned} \tag{1}$$

Where v_i and u_r are the weights assigned to input i (i = 1, . . . ,m) and output r (r = 1, . . . , s), respectively.Alirezaee and Afsharian [2] considered v_i and u_r as the ‘‘shadow price’’ of ith invested resource and rth product, respectively. $\sum_{r=1}^{s} u_r y_{rj}$ and $\sum_{i=1}^{m} v_i x_{ij}$ were considered as total revenue and total cost for DMUj, respectively. Hence, the second restriction of the profit for DMUj is introduced as follows

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0. j = 1,\dots,n \tag{2}$$

2.2. Wu et al. model

Although the initial data of DEA models and the method by Alirezaee and Afsharian [2] can be employed easily and calculated logically, it is not perfect because the new index for ranking DMUs is not stable and the rankings of the DMUs are also mutative. Wu et al. [3] modified the model into a unique solution problem. They indicate the Maximal Balance Index is a feasible and unique solution on the basis of assuring the DEA efficiency score of DMUp is EFF′p. Then the following model is proposed to compute the Maximal Balance Index

$$\begin{aligned}
\max\ & \Big(\sum_{i=1}^{m} v_j w_i - \sum_{r=1}^{s} u_r q_r\Big) \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ \ j=1,\dots,n; \\
& \sum_{i=1}^{m} v_i x_{ip} = 1, \\
& \sum_{r=1}^{s} u_r y_{rp} = EFF_p^{'} \\
& u_r, v_i \ge 0,\quad r=1,\dots,s;\ i=1,\dots,m
\end{aligned} \tag{3}$$

Where $w_i (i = 1, \dots, m)$ and $q_r (r = 1, \dots, s)$ are respectively the sum amount of ith input, rth output for all DMUs.

2.3. Dong Guo and Jie Wu's model

Following Wu's model, Guoan Wu [4] discusses undesirable outputs role in the mentioned models. This model points to proportion between inputs and outputs. Therefore, like inputs, some undesirable outputs are considered as an expense. Suppose there are n DMUs, each DMUj(j= 1, 2, . . . , n) consumes varying amounts of m inputs in the production of s desirable outputs and k undesirable outputs. The observed inputs, desirable outputs and undesirable outputs ofDMUj(j = 1, 2, . . . , n) are respectively denoted by xij(i = 1, . . . ,m), yrj(r = 1, . . . , s), btj(t = 1, . . . , k).

We consider the production possibility set (PPS) and characterize inputs and undesirable output.

$$\begin{aligned}
T = \Big\{ & (x,y,b) : \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i,\ \ i=1,\dots,m; \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_r,\ \ r=1,\dots,s; \\
& \sum_{j=1}^{n} \lambda_j b_t,\ \ t=1,\dots,k;\ \ \lambda_j \ge 0,\ \ j=1,\dots,n \Big\}.
\end{aligned} \tag{4}$$

According to the following model, the relative efficiency can be obtained as follow:

$$\begin{aligned}
\min\ & \theta_p \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_p x_{ip},\ \ i=1,\dots,m; \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp},\ \ r=1,\dots,s; \\
& \sum_{j=1}^{n} \lambda_j b_{tj} \le \theta_p b_{tp},\ \ t=1,\dots,k; \\
& \lambda_j \ge 0,\ \ j=1,\dots,n.
\end{aligned} \tag{5}$$

Model (5) is an input-oriented CCR model incorporating undesirable outputs. Correspondingly, its dual model is:

$$\begin{aligned}
&\max \sum_{r=1}^{5} u_r y_{rp} \\
&\text{s.t.} \quad \sum_{r=1}^{5} u_r y_{rj} - \sum_{i=1}^{m} u_i x_{ij} - \sum_{t=1}^{k} \eta_t b_{tj} \le 0, \forall j \\
&\quad \sum_{i=1}^{m} u_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp} = 1, \\
&\quad u_r, u_i \ge 0, \forall r \forall i \forall t.
\end{aligned} \tag{6}$$

The fractional form of the model (6) can be written as follow:

$$\begin{aligned}
&\max EFF_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp}} \\
&\text{s.t.} = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{k} \eta_t b_{tj}} \le 1, \forall j \\
&\quad u_r, v_i, \eta_t \ge 0, \forall r \forall i \forall t.
\end{aligned} \tag{7}$$

The Maximal Balance Index in incorporating undesirable outputs using Wu et al. can be computed by the following model:

$$\begin{aligned}
&\max\left(\sum_{r=1}^{m} v_i w_i + \sum_{t=1}^{k} \eta_t h_t - \sum_{i=1}^{s} u_r q_r\right) \\
&\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{k} \eta_t b_{tj} \le 0, \forall j \\
&\quad \sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp} = 1, \\
&\quad \sum_{r=1}^{s} u_r y_{rp} = EFF'_p \\
&\quad u_r, v_i, \eta_t \ge 0, \forall r \forall i \forall t
\end{aligned} \tag{8}$$

Where $w_i (i = 1, \ldots, m)$, q_r $(r = 1, \ldots, s)$ and h_t are respectively the sum amount of i_{th} input, r_{th} desirable output and t_{th} undesirable output for all DMUs. Since the Maximal Balance Index is unique, the corresponding rankings for all DMUs are stable. Therefore, regarding incorporating undesirable outputs,

DMU1 has a better rank than DMU2, if DMU1 is efficient butDMU2 is inefficient or both DMUs get the same efficiency score, and DMU1 obtains more negative quantity in the Maximal Balance Index.
Using εas the lower bound of factor weights, researchers indicated the inefficient DMUs which belong to weak parts of frontier. These lower bounds perturb the weak parts of frontier and in this manner weak efficient DMUs were appeared.

3. The presented method

Based on approach that is given by Daneshvar [5], the maximum value of weights for DMUs which are in weak part of frontier can be obtained without changing the PPS qualities and minimum value of each weight is considered as lower bound of weights. In this paper, these bounds are obtained by considering undesirable outputs and useful results are obtained with respect to previous models.. In order to illustrate the proposed model, we have used the following example.

Table 1 Data set for 7 DMUs

DMU	A	B	C	D	E	F	G
Input 1	4	7	8	4	2	2	2
Input 2	3	3	1	2	4	6	4
Desirable output	1	1	1	1	1	1	1
Undesirable output	0.7	0.3	0.8	0.8	0.15	0.15	0.4

Table 2 Efficiency score for Dong Guo and Jie Wu's

DMU	A	B	C	D	E	F	G
Efficiency	0.8571	0.9167	1	1	1	1	1

Table 3 The quantities of Maximal Balance Index

DMU	A	B	C	D	E	F	G
Maximal Balance Index	-	-	16	3.56	12	12	7.5

Table 4 Ranking of DMUs in classics model

DMU	A	B	C	D	E	F	G
Efficiency	6	5	1	4	2	2	3

4. Numerical Example

Table 1 shows Data set with two inputs and one desirable and one undesirable output.
Using table 1 and following model, the efficiency score for Dong Guo and Jie Wu's model can be computed. The results are shown in Table2.

$$\max EFF_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp}}$$

$$\text{s.t.} \quad = \frac{\sum_{r=1}^{5} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{k} \eta_t b_{tj}} \le 1, \forall j \qquad (9)$$

$$u_r, v_i, \eta_t \ge 0, \forall r \forall i \forall t.$$

As seen in Table 2, DMUA is the most inefficient DMU. DMUB ranks higher than DMUA. Other DMUs have the same efficiency score. Now, by applying Table1 and Table 2 and model (10) which is considered as Maximal Balance Index, Table 3 can be given.

Table 5 The optimal solution using $\frac{CCR}{\varepsilon}$ model

DMU	A	B	C	D	E	F	G
Efficiency	0.8541	0.8950	1	1	1	0.98	0.9957

Table 6 Maximal Balance Index of DMUs by $\frac{CCR}{\varepsilon}$ model

DMU	A	B	C	D	E	F	G
Maximal Balance Index	-	-	14.2945	3.5028	11.3800	-	-

$$\max \left(\sum_{r=1}^{m} v_i w_i + \sum_{t=1}^{k} \eta_t h_t - \sum_{i=1}^{s} u_r q_r\right)$$

$$\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{k} \eta_t b_{tj} \le 0, \forall j \quad (10)$$

$$\sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp} = 1, \qquad (10)$$

$$\sum_{r=1}^{s} u_r y_{rp} = EFF'_p$$

$$u_r, v_i, \eta_t \ge 0, \forall r \forall i \forall t$$

Using the results of Table 3, the ranking of DMUs can be written in table 4. As seen in Table4,DMUE and DMUF are same in ranking.

By considering $\varepsilon = 0.01$=, the optimal solution of the following model can be calculated easily. The results are shown in Table 5.

$$\max EFF_p = \frac{\sum_{r=1}^{s} u_r y_{rp}}{\sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp}}$$

$$\text{s.t.} \quad = \frac{\sum_{r=1}^{5} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{k} \eta_t b_{tj}} \le 1, \forall j \qquad (11)$$

$$u_r, v_i, \eta_t \ge 0.01, \forall r \forall i \forall t.$$

The results of Table 5 show that DMUA is the most inefficient DMU. Then DMUB, DMUF, DMUG get respectively higher rank. DMUC, DMUD, DMUF are in the same rank.

Table 7 Ranking of DMUs $\frac{CCR}{\varepsilon}$ model

DMU	A	B	C	D	E	F	G
Rank	7	6	1	3	2	5	4

By applying $\frac{CCR}{\varepsilon}$ model, the following model is given. The results are given in Table 6.

$$\max\left(\sum_{r=1}^{m} v_i w_i + \sum_{t=1}^{k} \eta_t h_t - \sum_{i=1}^{s} u_r q_r\right)$$

$$\text{s.t.} \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{t=1}^{k} \eta_t b_{tj} \le 0, \forall j \qquad (12)$$

$$\sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} \eta_t b_{tp} = 1,$$

$$\sum_{r=1}^{s} u_r y_{rp} = EFF'_p$$

$$u_r, v_i, \eta_t \ge 0.01, \forall r \forall i \forall t$$

The result of Table 5 is the complement of Table 5. The ranking of DMUs is determined in Table7.

Modification of Dong Guo and Jie Wu's model by applying PPS which is used by Dong Guo and Jie Wu's model [4], the following model is obtained to identify the maximum set of dual variables. We find that the undesirable outputs treat as inputs. On the other hands, we consider the profit of undesirable outputs is negative when undesirable outputs are jointly produced with desirable outputs.

$$
\begin{aligned}
\max\ & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ + \sum_{t=1}^{k} s_t^+ \\
\text{s.t.}\ & x_{ip} - \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = 0 \\
& b_{tp} - \sum_{j=1}^{n} \lambda_j b_{tj} - s_t^+ = 0 \\
& y_{rp} - \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = 0 \\
& \lambda_j \geq 0, s_i \geq 0, s_r^+ \geq 0, s_t^+ \geq 0 \\
& j = 1,2,\ldots,n \\
& i = 1,2,\ldots,m \\
& r = 1,2,\ldots,s \\
& t = 1,2,\ldots,k
\end{aligned} \tag{13}
$$

Table 8 Optimal solution of model (13), (14) and (15)

DMU	C	D	E	F	G
V_1^*	0.0833	0.1667	0.5	0.5	0.5
V_2^*	1	0.36	0.1818	0	0.1667
u_1^*	1	1	1	1	1
n_1^*	0.8	1.8182	6.6667	6.6667	0

The optimal values of u_r, v_i and n_t (r=1), (i=1, 2) and (t=1), for each DMU which is belonged to weak part frontier can be obtained via following model.

$$
\begin{aligned}
\max\ & v_i = v_i^* \\
\text{s.t.}\ & \sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} h_t b_p = 1 \\
& \sum_{r=1}^{s} u_r y_{rp} = 1 \\
& \sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} - \sum_{t=1}^{k} n_t b_{tp} \leq 0 \\
& v_i \geq 0, i = 1,2,\ldots,m \\
& u_r \geq 0, r = 1,2,\ldots,s \\
& n_t \geq 0, t = 1,2,\ldots,k
\end{aligned} \tag{14}
$$

$$\begin{aligned}
&\max u_r = u_r^* \\
&\text{s.t.} \quad \sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} h_t b_p = 1 \\
&\sum_{r=1}^{s} u_r y_{rp} = 1 \\
&\sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} - \sum_{t=1}^{k} n_t b_{tp} \le 0 \\
&v_i \ge 0, i = 1,2,...,m \\
&u_r \ge 0, r = 1,2,...,s \\
&n_t \ge 0, t = 1,2,...,k
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\max n_t = n_t^* \\
&\text{s.t.} \quad \sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{k} h_t b_p = 1 \\
&\sum_{r=1}^{s} u_r y_{rp} = 1 \\
&\sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} - \sum_{t=1}^{k} n_t b_{tp} \le 0 \\
&v_i \ge 0, i = 1,2,...,m \\
&u_r \ge 0, r = 1,2,...,s \\
&n_t \ge 0, t = 1,2,...,k
\end{aligned} \tag{16}$$

The results of model (14), (15) and (16) are given in table8

By considering Table8, the value of lower bound for v_1, v_2, u_1 and n_1 respectively can be obtained as follow:

$$\varepsilon_1^v = \min\{0.083, 0.1667, 0.5\} = 0.083$$
$$\varepsilon_2^v = \min\{1, 0.36, 0.81818, 0.1667\} = 0.667$$
$$\varepsilon_1^u = \min\{1\} = 1$$
$$\varepsilon_2^u = \min\{0.8, 1.8181, 6.6667\} = 0.8$$

By placing above values in model (6), the results are summarized in Table 9 and table 10.

Table 9 The efficiency score for modified model

DMU	A	B	C	D	E	F	G
Efficiency	0.8226	0.8903	1	1	1	0.9667	0.9800

Table 10 Maximal Balance Index of DMUs using modified model

DMU	A	B	C	D	E	F	G
Maximal Balance Index	-	-	13.4649	3.5124	11.0415	-	-

The results of ranking are summarized in Table11.

Table 11 Ranking of DMUs using modified model

DMU	A	B	C	D	E	F	G
Rank	7	6	1	3	2	5	4

Table 12 Efficiency score for different models

DMU	A	B	C	D	E	F	G
Efficiency (classic model)	0.8571	0.9167	1	1	1	1	1
Efficiency($\frac{CCR}{\varepsilon}$ model)	0.8541	0.8950	1	1	1	0.98	0.9957
Efficiency (modified model)	0.8226	0.8903	1	1	1	0.9667	0.9800

Table 13 Maximal Balance Index of DMUs for different models

DMU	A	B	C	D	E	F	G
Maximal Balance Index (classic model)	-	-	16	3.56	12	12	7.5
Maximal Balance Index ($\frac{CCR}{\varepsilon}$ model)	-	-	14.2945	3.5028	11.3800	-	-
Maximal Balance Index (modified model)	-	-	13.4649	3.5124	11.0415	-	-

By comparing the results of classic model and $\frac{CCR}{\varepsilon}$ model in above tables, we find that in classic model five DMUs are considered as efficient while in model the number of efficient DMUs is reduced to tree. In Maximal Balance Index of DMUs by classic model, the efficiency score of DMUE and DMUF is same while in $\frac{CCR}{\varepsilon}$ model they are different. As a result in classic model, ranking is not complete and $\frac{CCR}{\varepsilon}$ model give better results in comparison with classic model. By comparing $\frac{CCR}{\varepsilon}$ model with the modified model, it is clear that the final result of ranking is same for both methods, however by considering the result of efficiency score in these models we find that the modified model gives better results than $\frac{CCR}{\varepsilon}$ model. Because the efficiency scores are reduced. Also, maximal balance index of DMUC and DMUE has almost equal values. Therefore it causes to change the ranking of these DMUs.

5. Conclusions

To identify final ranking, classic model results same ranks for all DMUs. Hence, in classic model, ranking leads to failure. Perturbation of weak parts of frontier in $\frac{CCR}{\varepsilon}$ model implies weak efficient

DMUs and DMUs which compared with weak part of frontier to be considered as inefficient DMUs. But this method cannot make a remarkable changing in efficiency score. To solve this problem, we proposed a modified model that can change the efficiency score remarkably and improves the results.

References

[1] A. Charnes, W.W. Cooper, E. Rhodes, Measuring the efficiency of the decision making units, European Journal of Operational Research 2 (6) (1978) 429-444.

[2] M.R. Alirezaee, M. Afsharian, A complete ranking of DMUs using restrictions in DEA models, Applied Mathematics and Computation 189 (2) (2007) 1550–1559.

[3] Jie Wu, Feng Yang, Liang Liang, A modified complete ranking of DMUs using restrictions in DEA models, Applied Mathematics and Computation 217 (2) (2010) 745–751.

[4] Dong Guo, Jie Wu, A complete ranking of DMUs with undesirable outputs using restrictions in DEA models.

[5] S. Daneshvar, Facet Analysis in Data Envelopment Analysis, Thesis Submitted For Degree of Philosophy, Islamic Azad University (Science and Research Branch), 2002.