

A Solution Approach for Solving Fully Fuzzy Quadratic Programming Problems

Nemat Allah Taghi-Nezhad¹, Fatemeh Taleshian^{2}*

¹Department of Mathematics, Faculty of Basic Sciences, Gonbad Kavous University, Gonbad, Iran.

²Department of Mathematics, Shahrood University of Technology, Shahrood, Iran.

PAPER INFO	ABSTRACT
<p>Chronicle: Received: 05 October 2017 Accepted: 27 February 2018</p>	<p>Quadratic Programming has been widely applied to solve real-world problems. This paper describes a solution method for solving a special class of fuzzy quadratic programming problems with fuzziness in relations. Then the method is generalized to a more general fuzzy quadratic programming problem, where the cost coefficients, the matrix of the quadratic form, constraints coefficients, and the right-hand sides are all fuzzy numbers. Finally, some examples are taken to the utility of our proposed method.</p>
<p>Keywords: Fuzzy Number. Quadric Programming. Membership Function. Fuzzy Arithmetic. Fuzzy Constraint.</p>	

1. Introduction

In this paper, we deal with a Quadratic Programming problem. The classical Quadratic Programming problem is the problem of finding the minimum or maximum values of a quadratic function under constraints that are represented by linear inequality or equations. It's a mathematical modeling technique designed to optimize the usage of limited resources. It led to a number of interesting applications and the development of numerous useful results [9, 10]. In 2000, Beck and Teboulle considered nonconvex quadratic optimization problems with binary constraints and established a necessary global optimality condition for it [2]. They also presented an approach, which uses elementary arguments based on convex duality. In 2007, Jeyakumar et al. examined how global optimality of non-convex constrained optimization problems is related to Lagrange multiplier conditions [11]. In addition, they established Lagrange multiplier conditions for global optimality of general quadratic minimization problems with quadratic constraints. Xia obtained new sufficient optimality conditions for the nonconvex quadratic optimization problems with binary constraints by exploring local optimality conditions in 2009 [21]. In 2010, Glover et al. described a Diversification-Driven Tabu Search algorithm for solving unconstrained binary quadratic problems [8]. Also, Yang and Ruan presented a canonical duality theory for solving quadratic minimization problems subjected to either box or integer constraints [6]. In 2012, Sun et al. investigated the duality gap between the binary quadratic optimization problem and its semidefinite programming relaxation [17]. Frasch et al. addressed the

* Corresponding author
Email: fatemeh.taleshian@ut.ac.ir
DOI: 10.22105/jarie.2018.111797.1028

ubiquitous case in 2015 where the quadratic programming problems are strictly convex. The authors proposed a dual Newton strategy that exploits the block-bandedness similarly to an interior-point method [5]. Quadratic programming problem is mostly used in real problems. So the ambiguity and uncertainty are mostly presented in such optimization problems. Hence, the applicable tool fuzzy set theory is used to model these uncertainties in mathematical form. Fuzzy mathematical programming problems were quite studied in the specialized literature [3, 4, 12, 19, 20]. Also, fuzzy quadratic programming problem became interests of many researchers, and many studies were happened in this field. Abbasi Molai studied the quadratic programming problem with fuzzy relation inequality constraints in 2012 [16]. In 2014, Kochenberger et al. studied the unconstrained binary quadratic programming problem [13]. At this year, Gill and Wong studied active-set method for a generic quadratic programming problem with both equality and inequality constraints [7]. Also, Abbasi Molai studied the minimization problem of a quadratic objective function with the max-product fuzzy relation inequality constraints [15]. They use some properties of $n \times n$ real symmetric indefinite matrices, Cholesky's decomposition, and the least square technique to solve this problem. In 2017, Takapoui et al. proposed a fast optimization algorithm for approximately minimizing convex quadratic functions over the intersection of affine and separable constraints [18].

This paper is organized in six sections. In the next section, quadratic programming problem is defined. In Section 3, some necessary notations and definitions of fuzzy set theory and fuzzy arithmetic are given. In Section 4, fuzzy quadratic programming problem is defined in the first subsection, the solution method of solving this fuzzy quadratic programming problem is focused in the second subsection, and in the third subsection, we extend this idea for generalizing fuzzy quadratic programming problem. In Section 5, some numerical examples are presented to illustrate how to apply the contribution of this paper for solving such quadratic programming problems. Finally, we conclude in Section 6.

2. Quadratic Programming

The most typical quadratic programming problem is defined as follows [1]:

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i, i \in N_m, \\ x_j \geq 0 \end{cases} \end{aligned} \quad (1)$$

where $c = (c_1, c_2, \dots, c_n)$ and $b^T = (b_1, b_2, \dots, b_m)$ are called cost vector and right-hand side vector. $x = (x_1, x_2, \dots, x_n)^T$ is a vector of variables. Also, $A = [a_{ij}]_{m \times n}$, $i \in N_m$ and $j \in N_n$, is called a constraint matrix and $Q = [q_{ij}]_{m \times n}$ is called the matrix of quadratic form where $i \in N_n$ and $j \in N_n$. Using this notation, the problem can be modeled in matrix form as follows:

$$\begin{aligned} \text{Min } Z &= cx + \frac{1}{2} x^T Qx \\ \text{s. t. } &\begin{cases} Ax \leq b \\ x \geq 0 \end{cases} \end{aligned} \quad (2)$$

In this paper, the matrix $Q = [q_{ij}]_{m \times n}$, $i \in N_m$ and $j \in N_n$ is considered symmetric and positive semi-definite. Several studies have developed efficient and effective algorithms for solving quadratic programming when the value assigned to each parameter is a known constant. However, quadratic programming models usually are formulated to find some future course of action so the parameter

values used would be based on a prediction of future conditions, which inevitably involves some degree of uncertainty.

3. Fuzzy Numbers and Fuzzy Arithmetic

In this section some necessary definitions of fuzzy set theory which is taken from [9] are taken.

Definition 3.1: Let \mathbf{R} be the real line, then a fuzzy set \tilde{A} in \mathbf{R} is defined to be a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \mathbf{R}\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function for the fuzzy set. The membership function maps each element of \mathbf{R} to a membership value between 0 and 1.

Definition 3.2: The support of a fuzzy set \tilde{A} , is defined as follow:

$$\text{supp}(\tilde{A}) = \{x \in \mathbf{R} | \mu_{\tilde{A}}(x) > 0\}.$$

Definition 3.3: The core of a fuzzy set is the set of all points x in \mathbf{R} with $\mu_{\tilde{A}}(x) = 1$.

Definition 3.4: A fuzzy set \tilde{A} is called normal if its core is nonempty. In other words, there is at least one point $x \in \mathbf{R}$ with $\mu_{\tilde{A}}(x) = 1$.

Definition 3.5: The α -cut or α -level set of a fuzzy set is a crisp set defined as follows:

$$A_{\alpha} = \{x \in \mathbf{R} | \mu_{\tilde{A}}(x) > \alpha\}.$$

Definition 3.6: A fuzzy set \tilde{A} on \mathbf{R} is convex, if for any $x, y \in \mathbf{R}$ and $\lambda \in [0, 1]$, we have

$$\mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}.$$

Definition 3.7: A fuzzy number \tilde{A} is a fuzzy set on the real line that satisfies the condition of normality and convexity.

Definition 3.8: A fuzzy number \tilde{A} on \mathbf{R} is said to be triangular fuzzy number, if there exist real numbers and $l, r \geq 0$ such that

$$\tilde{A}(x) = \begin{cases} \frac{x}{l} + \frac{l-s}{l}, & x \in [s-l, s] \\ \frac{-x}{r} + \frac{s+r}{r}, & x \in [s, s+r] \\ 0, & o.w. \end{cases}.$$

We denote a triangular fuzzy number \tilde{A} by three real numbers s, l and r as $\langle s, l, r \rangle$ whose meaning are defined in Fig. 1. We also denote the set of all triangular fuzzy numbers with $\mathbf{F}(\mathbf{R})$.

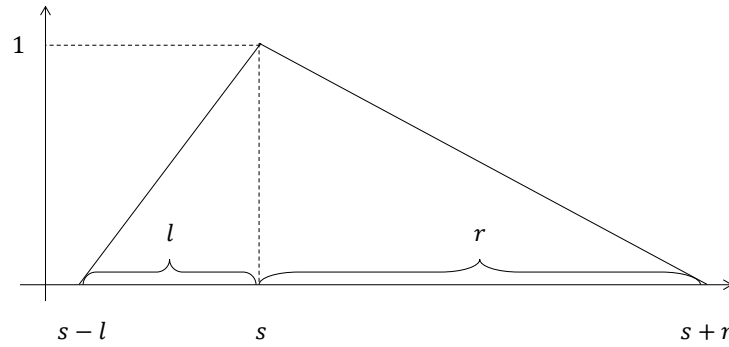


Fig. 1. Triangular Fuzzy Number.

Definition 3.9: Let $\tilde{a} = \langle s_a, l_a, r_a \rangle$ and $\tilde{b} = \langle s_b, l_b, r_b \rangle$ be two triangular numbers and $x \in \mathbf{R}$. Summation and multiplication of fuzzy numbers are defined as [22]:

$$x\tilde{a} = \begin{cases} \langle xs_a, xl_a, xr_a \rangle, & x \geq 0 \\ \langle xs_a, -xr_a, -xl_a \rangle, & x < 0 \end{cases}$$

$$\tilde{a} + \tilde{b} = \langle s_a + s_b, l_a + l_b, r_a + r_b \rangle$$

$$\tilde{a} - \tilde{b} = \langle s_a - s_b, l_a - r_b, r_a - l_b \rangle$$

$$\tilde{a} \leq \tilde{b} \text{ if and only if } s_a \leq s_b, s_a - l_a \leq s_b - l_b, s_a + r_a \leq s_b + r_b$$

Definition 3.10: We let $\tilde{0} = (0,0,0)$ as a zero triangular fuzzy number.

Remark 3.1: $\tilde{a} \geq \tilde{0}$ if and only if $s_a \geq 0, s_a - l_a \geq 0, s_a + r_a \geq 0$.

Remark 3.2: $\tilde{a} \leq \tilde{b}$ if and only if $-\tilde{a} \geq -\tilde{b}$.

4. Fuzzy Quadratic Programming

Consider the conventional quadratic programming problem (1). If some or all of the parameters were fuzzy numbers, the problem turns into the fuzzy quadratic programming problem. The most general type of this programming problem is formulated as follows:

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n \tilde{C}_j \tilde{X}_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \tilde{Q}_{ij} \tilde{X}_i \tilde{X}_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n \tilde{A}_{ij} \tilde{X}_j \leq \tilde{B}_i, & i \in N_m, \\ \tilde{X} \geq 0 \end{cases} \end{aligned} \quad (3)$$

where \tilde{A}_{ij} , \tilde{B}_i , \tilde{C}_j and \tilde{Q}_{ij} are fuzzy numbers, and \tilde{X}_j are variables whose states are fuzzy numbers ($i \in N_m, j \in N_n$). The operations of addition and multiplication are operations of fuzzy arithmetic. Here instead of discussing this general type, we consider two special cases of fuzzy quadratic programming problem as follows:

1. The quadratic programming problem that only the right-hand side parameters and constraint coefficient are triangular fuzzy numbers.

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n \tilde{A}_{ij} x_j \leq \tilde{B}_i, & i \in N_m. \\ x \geq 0 \end{cases} \end{aligned} \quad (4)$$

2. A more general fuzzy quadratic programming problem in which the cost coefficient, the matrix of the quadratic form, constraint coefficient, and right-hand side parameters are all triangular fuzzy numbers.

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n \tilde{C}_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \tilde{Q}_{ij} x_i x_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n \tilde{A}_{ij} x_j \leq \tilde{B}_i, & i \in N_m. \\ x \geq 0 \end{cases} \end{aligned} \quad (5)$$

In the following, a solving method for model (4) is represented. In addition, this method is extended to a more general fuzzy quadratic problem (5).

4.1. Quadratic Programming with Fuzziness in Relations

Consider the quadratic programming problem (4.2). Let $\tilde{A}_{ij} = \langle a_{ij}, l_{ij}, r_{ij} \rangle$ and $\tilde{B}_i = \langle b_i, u_i, v_i \rangle$ be triangular fuzzy numbers. According to Definition 3.8, the fuzzy quadratic programming (4.2) is rewritten as follows:

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n \langle a_{ij}, l_{ij}, r_{ij} \rangle x_j \leq \langle b_i, u_i, v_i \rangle, & i \in N_m. \\ x_j \geq 0, & j \in N_n \end{cases} \end{aligned} \quad (6)$$

Using the operations of fuzzy numbers, any fuzzy constraint $\sum_{j=1}^n \langle a_{ij}, l_{ij}, r_{ij} \rangle x_j \leq \langle b_i, u_i, v_i \rangle, i \in N_m$ can be transformed to the three crisp constraint as follows:

$$\begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i \\ \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq b_i - u_i \\ \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq b_i + v_i, & i \in N_m \end{cases}$$

Substituting these relations in Eq. (6), the fuzzy quadratic programming (4) is converted to the following conventional quadratic programming:

$$\begin{aligned}
 \min z = & \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\
 \text{s. t. } & \begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i, & i \in N_m \\ \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq b_i - u_i, & i \in N_m \\ \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq b_i + v_i, & i \in N_m \\ x_j \geq 0, & j \in N_n \end{cases}
 \end{aligned} \tag{7}$$

As it is clear, all numbers are involved in resent programming problem are real numbers. Hence, this classical quadratic programming problem can be easily solved using existing methods.

4.2. Fully Fuzzy Quadratic programming problem

In this section, we generalize above method to a more general fuzzy quadratic programming problem in which the cost coefficient, the matrix of quadratic form, constraint coefficient, and right-hand side parameters are all triangular fuzzy numbers.

$$\begin{aligned}
 \text{Min } Z = & \sum_{j=1}^n \tilde{C}_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \tilde{Q}_{ij} x_i x_j \\
 \text{s. t. } & \begin{cases} \sum_{j=1}^n \tilde{A}_{ij} x_j \leq \tilde{B}_i, & i \in N_m \\ x \geq 0 \end{cases}
 \end{aligned} \tag{8}$$

As it mentioned before, any triangular fuzzy number \tilde{A} can be represented as $\tilde{A} = \langle s, l, r \rangle$ where s, l and r are three real numbers. Using this representation, Problem (8) can then be rewritten as follows:

$$\begin{aligned}
 \text{Min } Z = & \sum_{j=1}^n \langle c_j, p_j, t_j \rangle x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \langle q_{ij}, s_{ij}, w_{ij} \rangle x_i x_j \\
 \text{s. t. } & \begin{cases} \sum_{j=1}^n \langle a_{ij}, l_{ij}, r_{ij} \rangle x_j \leq \langle b_i, u_i, v_i \rangle, & i \in N_m \\ x_j \geq 0, & i \in N_m \end{cases}
 \end{aligned} \tag{9}$$

where for all $i \in N_m$ and $j \in N_n$, $\tilde{Q}_{ij} = \langle q_{ij}, s_{ij}, w_{ij} \rangle$, $\tilde{A}_{ij} = \langle a_{ij}, l_{ij}, r_{ij} \rangle$, $\tilde{C}_j = \langle c_j, p_j, t_j \rangle$ and $\tilde{B}_i = \langle b_i, u_i, v_i \rangle$ are all triangular fuzzy numbers. Summation and multiplication are operations on fuzzy numbers, and \leq is the partial order, which is defined in Section 3. Then, the problem (9) can be rewritten as:

$$\begin{aligned} \text{Min } Z &= \sum_{j=1}^n \langle c_j, p_j, t_j \rangle x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \langle q_{ij}, s_{ij}, w_{ij} \rangle x_i x_j \\ \text{s. t. } &\begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i, & i \in N_m, \\ \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq (b_i - u_i), & i \in N_m, \\ \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq (b_i + v_i), & i \in N_m, \\ x \geq 0 \end{cases} \end{aligned} \tag{10}$$

In the above optimization problem since the coefficients of the objective function are still fuzzy numbers, the optimal solution and so the derived objective value are fuzzy numbers as well. Without loss of generality, let $\mu_{\tilde{c}_j}$ and $\mu_{\tilde{Q}_{ij}}$, denote the membership function of \tilde{C}_j and \tilde{Q}_{ij} , respectively:

$$\begin{aligned} \mu_{\tilde{c}_j} &= \left\{ \left(C_j, \mu_{\tilde{c}_j}(C_j) \right) \mid C_j \in S(\tilde{C}_j) \right\} \\ \mu_{\tilde{Q}_{ij}} &= \left\{ \left(q_{ij}, \mu_{\tilde{Q}_{ij}}(q_{ij}) \right) \mid q_{ij} \in S(\tilde{Q}_{ij}) \right\}, \end{aligned}$$

where $S(\tilde{C}_j)$ and $S(\tilde{Q}_{ij})$ are support sets of \tilde{C}_j and \tilde{Q}_{ij} , respectively. In this paper, we are interested in deriving the membership function of the objective value \tilde{z} . Since \tilde{z} is a fuzzy number rather than a crisp number, we apply Zadeh's extension principle to transform the problem into a family of conventional quadratic programs to solve. Based on the extension principle, the membership function $\mu_{\tilde{z}}$, can be defined as:

$$\sup_{a,b,c} \text{Min} \left\{ \mu_{\tilde{Q}_{ij}}, \mu_{\tilde{c}_j}, \forall i, j \mid z = Z(a, b, c, q) \right\}, \tag{11}$$

where $Z(a, b, c, q)$ is the function of the conventional quadratic program that is defined in Model (1). Intuitively, to find the membership function $\mu_{\tilde{z}}$, it is sufficient to find the right shape function and the left shape function of $\mu_{\tilde{z}}$, which is equivalent to find the upper bound of the objective value z_α^u and the lower bound of the objective z_α^l at specific α level. Since z_α^u is the maximum of $Z(a, b, c)$ and z_α^l is the minimum of $Z(a, b, c)$, they can be expressed as:

$$z_\alpha^u = \text{Max} \left\{ Z(a, b, c, q) \mid \begin{matrix} (\tilde{Q}_{ij})_\alpha^l \leq q_{ij} \leq (\tilde{Q}_{ij})_\alpha^u, \\ (\tilde{C}_j)_\alpha^l \leq c_j \leq (\tilde{C}_j)_\alpha^u, \\ \forall i, j \end{matrix} \right\} \tag{12}$$

$$z_\alpha^l = \text{Min} \left\{ Z(a, b, c, q) \mid \begin{matrix} (\tilde{Q}_{ij})_\alpha^l \leq q_{ij} \leq (\tilde{Q}_{ij})_\alpha^u, \\ (\tilde{C}_j)_\alpha^l \leq c_j \leq (\tilde{C}_j)_\alpha^u, \\ \forall i, j \end{matrix} \right\}. \tag{13}$$

Clearly, different values of q_{ij} and c_j produce different objective values. To find the interval of the objective value at specific α value, it is sufficient to find the upper bound and lower bound of the objective values of (4.7). From (4.9), the values of q_{ij} and c_j , that attain the largest value for z_α^u , can be determined from the following two-level mathematical programming model:

$$\begin{aligned}
 & \text{Max} \\
 & z_{\alpha}^u = (\tilde{Q}_{ij})_{\alpha}^l \leq q_{ij} \\
 & \leq (\tilde{Q}_{ij})_{\alpha}^u \\
 & (\tilde{C}_j)_{\alpha}^l \leq c_j \leq (\tilde{C}_j)_{\alpha}^u \\
 & \forall i, j
 \end{aligned}
 \quad
 \begin{aligned}
 & \text{Min } Z = \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\
 & \text{s. t. } \left\{ \begin{aligned} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i \in N_m \\ & \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq (b_i - u_i), \quad i \in N_m \\ & \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq (b_i + v_i), \quad i \in N_m \\ & x \geq 0 \end{aligned} \right.
 \end{aligned}
 \tag{14}$$

The objective value z_{α}^u , is the upper bound of the objective values for model (10). By the same token, to find the values of q_{ij} and c_j that produce the smallest objective value z_{α}^l , a two-level mathematical program is formulated by replacing the outer-level program of Model (14) from ‘‘Min’’ to ‘‘Max’’:

$$\begin{aligned}
 & \text{Min} \\
 & z_{\alpha}^l = (\tilde{Q}_{ij})_{\alpha}^l \leq q_{ij} \leq (\tilde{Q}_{ij})_{\alpha}^u \\
 & (\tilde{C}_j)_{\alpha}^l \leq c_j \leq (\tilde{C}_j)_{\alpha}^u \\
 & \forall i, j
 \end{aligned}
 \quad
 \begin{aligned}
 & \text{Min } Z = \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \\
 & \text{s. t. } \left\{ \begin{aligned} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i \in N_m \\ & \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq (b_i - u_i), \quad i \in N_m \\ & \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq (b_i + v_i), \quad i \in N_m \\ & x \geq 0 \end{aligned} \right.
 \end{aligned}
 \tag{15}$$

Clearly, for any $x_i, x_j \geq 0$ we have:

$$\begin{aligned}
 & (\tilde{Q}_{ij})_{\alpha}^l x_i x_j \leq q_{ij} x_i x_j \leq (\tilde{Q}_{ij})_{\alpha}^u x_i x_j \\
 & (\tilde{C}_j)_{\alpha}^l x_j \leq c_j x_j \leq (\tilde{C}_j)_{\alpha}^u x_j.
 \end{aligned}$$

In other words, we have:

$$\sum_{j=1}^n (\tilde{C}_j)_{\alpha}^l x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\tilde{Q}_{ij})_{\alpha}^l x_i x_j \leq \sum_{j=1}^n c_j x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n q_{ij} x_i x_j \leq \sum_{j=1}^n (\tilde{C}_j)_{\alpha}^u x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\tilde{Q}_{ij})_{\alpha}^u x_i x_j$$

To derive the maximum objective value in (14), the parameters q_{ij} and c_j must set to their upper bounds $(\tilde{Q}_{ij})_{\alpha}^u$ and $(\tilde{C}_j)_{\alpha}^u$. Simultaneously, to search for the minimum objective value in (15), we need to set all q_{ij} and c_j to their lower bounds $(\tilde{Q}_{ij})_{\alpha}^l$ and $(\tilde{C}_j)_{\alpha}^l$, respectively at the same time. Models (14) and (15) can be respectively reformulated as:

$$\begin{aligned}
 Z_{\alpha}^u = \text{Min } Z &= \sum_{j=1}^n (\tilde{C}_j)_{\alpha}^u x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\tilde{Q}_{ij})_{\alpha}^u x_i x_j \\
 \text{s.t. } &\begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i, & i \in N_m \\ \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq (b_i - u_i), & i \in N_m \\ \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq (b_i + v_i), & i \in N_m \\ x \geq 0, \end{cases}
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 Z_{\alpha}^l = \text{Min } Z &= \sum_{j=1}^n (\tilde{C}_j)_{\alpha}^l x_j + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (\tilde{Q}_{ij})_{\alpha}^l x_i x_j \\
 \text{s.t. } &\begin{cases} \sum_{j=1}^n a_{ij} x_j \leq b_i, & i \in N_m \\ \sum_{j=1}^n (a_{ij} - l_{ij}) x_j \leq (b_i - u_i), & i \in N_m \\ \sum_{j=1}^n (a_{ij} + r_{ij}) x_j \leq (b_i + v_i), & i \in N_m \\ x \geq 0, \end{cases}
 \end{aligned} \tag{17}$$

These are pairs of one-level mathematical programs that clearly express the bounds of the objective values at a specific α value. In these conventional quadratic programming problems, the objective functions are convex functions and constraints are linear. Therefore, the derived objective values are global optimum solutions. The optimal solutions Z_{α}^l and Z_{α}^u are the lower bound and the upper bound of the objective values of the fuzzy quadratic programming problems, respectively. Z_{α}^l and Z_{α}^u together constitute the interval $[Z_{\alpha}^l, Z_{\alpha}^u]$ that the objective values of the fuzzy quadratic program lie. In the next section, by using two numerical examples we illustrate the utility of this paper.

5. Numerical Examples

In this section, the applicability of our proposed method is demonstrated by solving two numerical examples.

Example 1: In the following, a quadratic programming problem is considered in which the constraint coefficient and the right-hand sides parameters are triangular fuzzy parameters:

$$\begin{aligned}
 \text{Min } Z &= 2x_1 + x_2 + 2x_1^2 + x_1x_2 + 2x_2^2 \\
 \text{s.t. } &\begin{cases} \langle 6, 2, 1.5 \rangle x_1 + \langle 8, 2.5, 1 \rangle x_2 \leq \langle 5, 1, 0.5 \rangle \\ \langle 5, 1, 1 \rangle x_1 + \langle 2, 0.5, 1 \rangle x_2 \leq \langle 7, 2, 1.5 \rangle \\ x_1, x_2 \geq 0 \end{cases}
 \end{aligned} \tag{18}$$

According to proposed approach in Section 4, the above quadratic programming problem can be transform in to fallowing form:

$$\begin{aligned} \text{Min } Z &= 2x_1 + x_2 + 2x_1^2 + x_1x_2 + 2x_2^2 \\ \text{s. t. } &\begin{cases} 6x_1 + 8x_2 \leq 5 \\ 5x_1 + 2x_2 \leq 7 \\ 4x_1 + 5.5x_2 \leq 4 \\ 7.5x_1 + 9x_2 \leq 5.5 \\ 4x_1 + 1.5x_2 \leq 5 \\ 6x_1 + 3x_2 \leq 8.5 \\ x_1, x_2 \geq 0 \end{cases} \end{aligned} \quad (19)$$

where parameter values are all known constant. Thus, this model is a conventional quadratic programming problem. By solving this problem with SQP algorithm, the global optimum solution is obtained as:

$$x_1 = 0, \quad x_2 = 0.$$

The value of the objective function is also achieved $Z^* = 0$.

Example 2: Now suppose a quadratic programming problem with fuzzy objective function and fuzzy constraints as follows:

$$\begin{aligned} \text{Min } \tilde{Z} &= \langle -5, 1, 1 \rangle x_1 + \langle 1.5, 0.5, 0.5 \rangle x_2 + \frac{1}{2} (\langle 6, 2, 2 \rangle x_1^2 + \langle -4, 2, 2 \rangle x_1x_2 + \langle 4, 2, 2 \rangle x_2^2) \\ \text{s. t. } &\begin{cases} 1x_1 + \langle 1, 0.5, 0.5 \rangle x_2 \leq \langle 2, 1, 1 \rangle \\ \langle 2, 1, 1 \rangle x_1 + \langle -1, 1, 0.5 \rangle x_2 \leq \langle 4, 1, 1 \rangle \\ x_1, x_2 \geq 0 \end{cases} \end{aligned} \quad (20)$$

According to Models (16) and (17), the upper and lower bounds of \tilde{Z} at possibility level α can be solved as:

$$\begin{aligned} Z_\alpha^u &= \text{Min } (-4 - \alpha)x_1 + \left(2 - \frac{\alpha}{2}\right)x_2 + (4 - \alpha)x_1^2 + (-\alpha - 1)x_1x_2 \\ &\quad + (3 - \alpha)x_2^2 \\ \text{s. t. } &\begin{cases} x_1 + x_2 \leq 2 \\ x_1 + 0.5x_2 \leq 1 \\ x_1 + 1.5x_2 \leq 3 \\ 2x_1 - x_2 \leq 4 \\ x_1 - 2x_2 \leq 3 \\ 3x_1 - 0.5x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases} \end{aligned} \quad (21)$$

$$\begin{aligned} Z_\alpha^l &= \text{min } (\alpha - 6)x_1 + \left(1 + \frac{\alpha}{2}\right)x_2 + (2 + \alpha)x_1^2 + (\alpha - 3)x_1x_2 \\ &\quad + (1 + \alpha)x_2^2 \\ \text{s. t. } &\begin{cases} x_1 + x_2 \leq 2 \\ x_1 + 0.5x_2 \leq 1 \\ x_1 + 1.5x_2 \leq 3 \\ 2x_1 - x_2 \leq 4 \\ x_1 - 2x_2 \leq 3 \\ 3x_1 - 0.5x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases} \end{aligned} \quad (22)$$

These two models are pairs of traditional quadratic programs. Table 1 lists the α -cuts of the objective value at six distinct α values: 0, 0.2, ..., 1.

Table 1. Lists the α -cuts of the objective value.

α	0	0.2	0.4	0.6	0.8	1
z_{α}^l	-	-	-	-	-	-
	4.8	3.7	3.3	2.9	2.5	2.1
z_{α}^u	-1	-	-	-	-	-
		1.2	1.3	1.6	1.8	2.1

Thus, the optimal objective value is equivalent to:

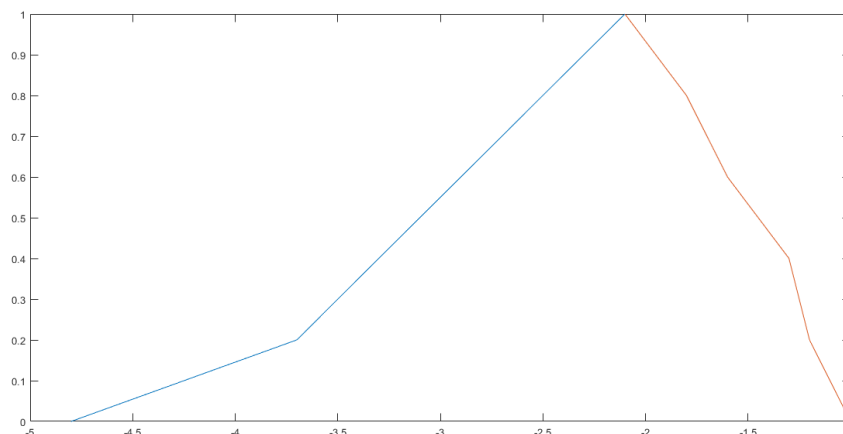


Fig. 2. Optimal objective value.

6. Conclusion

Quadratic programming problem is an important field in operation research and is mostly used to optimizing real words problems. In this paper, we considered a special class of quadratic programming problem with fuzziness in constraints coefficients. In addition, we considered all fuzzy numbers in triangular form. Then, we presented a new method using fuzzy operations to solve the mentioned problem. Finally, by combining our idea with Liu’s method, we extended our method for solving a more general quadratic programming problem with fuzziness in both constraints and objective function coefficients. We tried to extend this method to situations where parameters are trapezoidal or LR fuzzy numbers. Also, we hope to extend our presented method for problems with fuzzy parameters and fuzzy variables.

Reference

- [1] Bazaraa, M. S., Sherali, H. D., & Shetty, C. M. (2013). *Nonlinear programming: theory and algorithms*. John Wiley & Sons.
- [2] Beck, A., & Teboulle, M. (2000). Global Optimality conditions for quadratic optimization problems with binary constraints. *SIAM journal on optimization*, 11(1), 179-188.
- [3] Das, S. K., & Mandal, T. (2017). A new model for solving fuzzy linear fractional programming problem with ranking function. *Journal of applied research on industrial engineering*, 4(2), 89-96.
- [4] Ebrahimnejad, A., & LUIS, J. (2018). *Fuzzy Sets-Based Methods and Techniques for Modern Analytics*. Springer International Publisher.
- [5] Frasch, J. V., Sager, S., & Diehl, M. (2015). A parallel quadratic programming method for dynamic optimization problems. *Mathematical programming computation*, 7(3), 289-329.
- [6] Gao, D. Y., & Ruan, N. (2010). Solutions to quadratic minimization problems with box and integer constraints. *Journal of global optimization*, 47(3), 463-484.

- [7] Gill, P. E., & Wong, E. (2015). Methods for convex and general quadratic programming. *Mathematical programming computation*, 7(1), 71-112.
- [8] Glover, F., Lü, Z., & Hao, J. K. (2010). Diversification-driven tabu search for unconstrained binary quadratic problems. *4OR*, 8(3), 239-253.
- [9] Goodarzi, F. K., Taghinezhad, N. A., & Nasser, S. H. (2014). A new fuzzy approach to solve a novel model of open shop scheduling problem. *University politehnica of bucharest scientific bulletin-series a-applied mathematics and physics*, 76(3), 199-210.
- [10] Hock, W., & Schittkowski, K. (1980). Test examples for nonlinear programming codes. *Journal of optimization theory and applications*, 30(1), 127-129.
- [11] Jeyakumar, V., Rubinov, A. M., & Wu, Z. Y. (2007). Non-convex quadratic minimization problems with quadratic constraints: global optimality conditions. *Mathematical programming*, 110(3), 521-541.
- [12] Kao, C., & Liu, S. T. (2000). Fuzzy efficiency measures in data envelopment analysis. *Fuzzy sets and systems*, 113(3), 427-437.
- [13] Kochenberger, G., Hao, J. K., Glover, F., Lewis, M., Lü, Z., Wang, H., & Wang, Y. (2014). The unconstrained binary quadratic programming problem: a survey. *Journal of combinatorial optimization*, 28(1), 58-81.
- [14] Maros, I., & Mészáros, C. (1999). A repository of convex quadratic programming problems. *Optimization methods and software*, 11(1-4), 671-681.
- [15] Molai, A. A. (2014). A new algorithm for resolution of the quadratic programming problem with fuzzy relation inequality constraints. *Computers & industrial engineering*, 72, 306-314.
- [16] Molai, A. A. (2012). The quadratic programming problem with fuzzy relation inequality constraints. *Computers & industrial engineering*, 62(1), 256-263.
- [17] Sun, X. L., Liu, C. L., Li, D., & Gao, J. J. (2012). On duality gap in binary quadratic programming. *Journal of global optimization*, 53(2), 255-269.
- [18] Takapoui, R., Moehle, N., Boyd, S., & Bemporad, A. (2017). A simple effective heuristic for embedded mixed-integer quadratic programming. *International journal of control*, 1-11.
- [19] Taleshian, F., & Fathali, J. (2016). A Mathematical Model for Fuzzy-Median Problem with Fuzzy Weights and Variables. *Advances in operations research*. <http://dx.doi.org/10.1155/2016/7590492>
- [20] Taleshian, F., Fathali, J. & Taghi-Nezhad, N. A. (in press). Fuzzy majority algorithms for the 1-median and 2-median problems on a fuzzy tree. *Fuzzy Information and Engineering*.
- [21] Xia, Y. (2009). New optimality conditions for quadratic optimization problems with binary constraints. *Optimization letters*, 3(2), 253-263.
- [22] Zimmermann, H.J. (1986) Fuzzy Set Theory and Mathematical Programming. *Fuzzy sets theory and applications*, 99 -114.