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On Solving Capacitated Transportation Problem

Kolsoom Ahmadi*
Department of Mathematics, Kurdistan University, Sanandaj, Iran.

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ABSTRACT

We present a modification of three existing methods for finding a basic feasible solution for capacitated transportation problem. To obtain an optimal solution, the simplex algorithm for bounded variables is applied. Special properties of transportation problem help us to operate each step of simplex algorithm directly on the transportation tableau. At last, numerical examples are represented to illustrate our method.

1. Introduction

The feasible shipment of the products to wholesalers or to warehouses is a common problem in companies. Such problem is called a transportation problem, which is a special case of the linear programming problems. The general model corresponds to the classical transportation problem, comprises of the objective function, supply constraints, demand constraints, and non-negativity constraints. However, if the decision variables which are the amounts of shipment have capacity constraints for various reasons such as capacity of tracks, warehouse capacity, etc., a capacitated transportation model is used.

Capacitated transportation model occurs frequently in applications and it is important to be able to handle the capacity constraints efficiently. This kind of problem can be solved by simplex algorithm for bounded variables [1]. Various authors have studied balanced capacitated transportation problems. Kassay proposed an operator method for solving capacitated transportation problem [7]. Hassain and Zemel studied probabilistic analysis of capacitated transportation problem [6]. They assumed that the capacities are random variables, and proved asymptotic conditions on the supplies and demands which assure that a feasible solution exists almost surely. For studying other researches in this field, one can refer to [2, 8, 13].

* Corresponding author
E-mail address: kolsoom.ahmadi@gmail.com
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Dahiya and Verma considered a class of the capacitated transportation problems with bounds on total availabilities at sources and total destination requirements [3]. They obtained an equivalent balanced capacitated transportation problem for this class of problems. There are several methods for constructing initial basic feasible solutions for transportation problem, i.e. allocating $m+n-1$ basic variables which satisfy all constraint equations (e.g. [4, 5, 10, 11]). In capacitated transportation, we have some extra capacity constraints, and the mentioned methods should be modified in order to encase these constraints. We present a modification of three well known methods. In these modified methods, size of the problem does not change, unlike [9] which added a row and a column to the transportation tableau.

In Section 2, we describe the transportation model with bounds on variables and we give a necessary and sufficient condition which assures the feasibility of problem. Section 3 consists of three modified algorithms for constructing an initial basic feasible solution of the problem. Some examples show performance of these algorithms. In Section 4, we explain transportation simplex algorithm for bounded variables to solve one example of the previous section. Comparison the proposed algorithm with an existing algorithm is presented in Section 5. Section 6 contains a short conclusion.

2. Balanced Capacitated Transportation Model

Consider the following balanced transportation:

$$\begin{aligned} &\textit{Minimize} \quad && z=\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} && \\ &\textit{subject to} && \sum_{j\in J} x_{ij}=s_i && i\in I \\ & && \sum_{i\in I} x_{ij}=d_j && j\in J \\ & && l_{ij}\le x_{ij}\le u_{ij} && i\in I, j\in J, \end{aligned} \tag{1}$$

where $I=\{1,2,\dots,m\}$ is the index set of m sources, $J=\{1,2,\dots,n\}$ is the index set of n destinations, x_{ij} stands for the quantity transported from source i to destination j, c_{ij} is the cost of transporting one unit between source i and destination j, $s_i\ge 0$ $(i\in I)$ is the supply of source i, $d_j\ge 0$ $(j\in J)$ is the demand of destination j, $l_{ij}\ge 0$ and we assume $\sum_{i\in I} s_i=\sum_{j\in J} d_j$. We also have $\sum_{i\in I} l_{ij}\le d_j$, $\sum_{j\in J} l_{ij}\le s_i$, $\sum_{i\in I} u_{ij}\ge d_j$, $\sum_{j\in J} u_{ij}\ge s_i$ to make the problem consistent.

In order to solve problem (1), consider the equivalent transportation problem (2) as follows:

$$\begin{aligned} &\textit{Minimize} \quad && z=\sum_{i\in I}\sum_{j\in J} c_{ij}t_{ij}+A && \\ &\textit{subject to} && \sum_{j\in J} t_{ij}=s'_i && i\in I \\ & && \sum_{i\in I} t_{ij}=d'_j && j\in J \\ & && 0\le t_{ij}\le u_{ij}-l_{ij}, && i\in I, j\in J, \end{aligned} \tag{2}$$

where $A=\sum_{i\in I}\sum_{j\in J} c_{ij}l_{ij}$, $s'_i=s_i-\sum_{j\in J} l_{ij}$ and $d'_j=d_j-\sum_{i\in I} l_{ij}$.

Clearly, corresponding to every feasible solution t_{ij} of problem (2), there exists a feasible solution $x_{ij} = t_{ij} + l_{ij}$ of problem (1), and corresponding to every feasible solution x_{ij} of (2), there exists a feasible solution $t_{ij} = x_{ij} - l_{ij}$ of problem (2). The value of the objective function of problem (1) at a feasible solution is equal to the value of the objective function of (2) at its corresponding feasible solution and conversely. Finally, there is a one-to-one correspondence between optimal solutions to (1) and optimal solutions to (2). Hence, instead of problem (1), we can solve problem (3) as follows:

$$\begin{array}{lll} \text{Minimize} & z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} & \\ \text{subject to} & \sum_{j \in J} x_{ij} = s_i & i \in I \\ & \sum_{i \in I} x_{ij} = d_j & j \in J \\ & 0 \le x_{ij} \le u_{ij} & i \in I, j \in J. \end{array} \tag{3}$$

Lemma 1 provides a necessary and sufficient condition for problem (3) to be feasible.

Lemma 1. Problem (3) is feasible if and only if $\frac{s_i d_j}{d} \le u_{ij}$ for all $i \in I$ and $j \in J$, where $d = \sum_{i \in I} s_i = \sum_{j \in J} d_j$.

proof. Suppose that $\frac{s_i d_j}{d} \le u_{ij}$ for all $i \in I$ and $j \in J$. Set $x_{ij} = \frac{s_i d_j}{d}$, we have $\sum_{j \in J} x_{ij} = s_i$ for all $i \in I$, $\sum_{i \in I} x_{ij} = d_j$ for all $j \in J$ and $0 \le x_{ij} \le u_{ij}$, so x_{ij} is a feasible solution for (3). Conversely, suppose by contradiction that $\exists (i, j) \in I \times J,\ s.t.\ \frac{s_i d_j}{d} > u_{ij}$. We have $d_j = \sum_{i \in I} \frac{s_i d_j}{d} > \sum_{i \in I} u_{ij} \ge d_j$, and this is a violation. So, we should have $\frac{s_i d_j}{d} \le u_{ij}$ for all $i \in I$ and $j \in J$.

Consider a feasible (3) in which the condition stated in lemma 1 is hold. Since (3) is bounded, there exists at least one optimal solution [1]. We use simplex algorithm for bounded variables to find this optimal solution.

3. Finding an Initial Basic Feasible Solution

To start the transportation simplex, a Basic Feasible Solution (BFS) is needed. There are several methods for obtaining a starting BFS of general transportation in which there aren't upper bounds on variables. In what follows we modify three of the existing methods to make them suitable for capacitated transportation. For a definition of basic feasible solutions of bounded linear programming, see [1].

3.1. Modified Northwest Corner Method (MNCM)

This algorithm has two phases:

Phase 1: The algorithm begins with $i=1$, $j=1$, $\hat{s}_i = s_i$, $\hat{d}_j = d_j$.

Step 1:

$$x_{ij} = \min\{\hat{s}_i, \hat{d}_j, u_{ij}\}.$$
$$\hat{s}_i \to \hat{s}_i - x_{ij}.$$
$$\hat{d}_j \to \hat{d}_j - x_{ij}.$$

Step 2:

Case i: $x_{ij} = \hat{s}_i \le \{\hat{d}_j, u_{ij}\}$. $i \to i+1$, $j \to j$.

Case ii: $x_{ij} = \hat{d}_j \le \{\hat{s}_i, u_{ij}\}$. $i \to i$, $j \to j+1$.

Case iii: $x_{ij} = u_{ij} < \{\hat{s}_i, \hat{d}_j\}$. In this case, x_{ij} will be non-basic at its upper bound and we will have two new cells. $i \to i+1$, $j \to j$ and $i \to i$, $j \to j+1$.

If $i = m$, $j = n$ go to Step 3, otherwise go back to Step 1 (In case iii, the algorithm is repeated twice).

Step 3: $x_{mn} = \min\{\hat{s}_m, \hat{d}_n\}$ (Note that $\hat{s}_m = \hat{d}_n$ since the problem is balanced).

Remark 1: If in one iteration, a row and a column are both satisfied, we move to one of the cells $(i, j+1)$ or $(i+1, j)$ arbitrarily.

Remark 2: If in one iteration of the algorithm, we have one of these three cases: $\hat{s}_i = \hat{d}_j = 0$, $\hat{s}_i = u_{ij}$ or $\hat{d}_j = u_{ij}$, degeneracy occurs.

When Phase 1 terminates, the last obtaining variable is $x_{mn} = \hat{s}_m = \hat{d}_n$. If $x_{mn} \le u_{mn}$, current solution is feasible otherwise we should go to Phase 2.

Phase 2: Suppose that $x_{mn} > u_{mn}$, this causes infeasibility and we should exit x_{mn} from the basis. There exists a unique cycle for the cell (m,n); all corners of this cycle are basic except for one. We update the values of the variables in this cycle by $\Delta = x_{mn} - u_{mn}$ according to the signs of the cells in the cycle (sign of the cell (m,n) is negative; this sign changes to positive for the adjacent cell and so on). After updating the table, if feasibility holds $(0 \le x_{ij} \le u_{ij})$, we arrive at a basic feasible solution, if not we should repeat the above process. This process is similar to dual simplex method and since problem (3) is feasible and finite, at last a basic feasible solution will be obtained.

Remark 3: The above process can be done similarly if in one step $x_{ij} \leq 0$. The only difference is that $\Delta = x_{ij}$ and the sign of the cells in the cycle starts from positive, since we should increase x_{ij} to zero. To illustrate the algorithm, we operate it in two numerical examples. The first example goes to Phase 2 and the second one stops at Phase 1.

Example 1: Consider the following capacitated transportation problem:

$$\begin{aligned} &\min \quad z = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \\ &\textit{subject to} \quad \sum_{j=1}^{4} x_{1j} = 15, \ \sum_{j=1}^{4} x_{2j} = 25, \ \sum_{j=1}^{4} x_{3j} = 40, \\ &\sum_{i=1}^{3} x_{i1} = 10, \ \sum_{i=1}^{3} x_{i2} = 23, \ \sum_{i=1}^{3} x_{i3} = 22, \ \sum_{i=1}^{3} x_{i4} = 25, \end{aligned}$$

where

$$\begin{aligned} &4 \leq x_{11} \leq 10 \ , \ 5 \leq x_{12} \leq 15 \ , \ 1 \leq x_{13} \leq 12 \ , \ 3 \leq x_{14} \leq 14 \ , \ 1 \leq x_{21} \leq 15 \ , \ 5 \leq x_{22} \leq 10, \\ &3 \leq x_{23} \leq 12 \ , \ 0 \leq x_{24} \leq 10 \ , \ 0 \leq x_{31} \leq 8 \ , \ 2 \leq x_{32} \leq 14 \ , \ 3 \leq x_{33} \leq 17 \ , \ 0 \leq x_{34} \leq 15. \end{aligned}$$

Values of $c_{ij}, u_{ij} - l_{ij}, s'_i, d'_j$ are shown in Table 1 (c_{ij} and $u_{ij} - l_{ij}$ are in the left corner and right corner of the cells, respectively).

Table 1. *Values of* $c_{ij}, u_{ij} - l_{ij}, s'_i, d'_j$.

	1		2		3		4		Supply
1	10	6	12	10	13	11	8	11	2
2	15	14	18	5	12	9	16	10	16
3	17	8	16	12	13	14	14	15	35
Demands	5		11		15		22		

Applying (MNCM) yields the following iterations:

Iteration 1: $x_{11} = \min\{2,5,6\} = 2$,$\hat{s}_1 = 2-2 = 0$,$\hat{d}_1 = 5-2 = 3$. The next cell is (2,1).

Iteration 2: $x_{21} = \min\{16,3,14\} = 3$,$\hat{s}_2 = 16-3 = 13$,$\hat{d}_1 = 3-3 = 0$. The next cell is (2,2).

Iteration 3: $x_{22} = \min\{13,11,5\} = 5$,$\hat{s}_2 = 13-5 = 8$,$\hat{d}_2 = 11-5 = 6$. The next cells are (2,3) and (3,2).

Iteration 4.1: $x_{23} = \min\{8,15,9\} = 8$,$\hat{s}_2 = 8-8 = 0$,$\hat{d}_3 = 15-8 = 7$. The next cell is (3,3).

Iteration 4.2: $x_{32} = \min\{35,6,12\} = 6$,$\hat{s}_3 = 35-6 = 29$,$\hat{d}_2 = 6-6 = 0$. The next cell is (3,3).

Iteration 5: $x_{33} = \min\{29,7,14\} = 7$,$\hat{s}_3 = 29-7 = 22$,$\hat{d}_3 = 7-7 = 0$. The last cell is (3,4).

Iteration 6: $x_{34} = \min\{22,22\} = 22$,$\hat{s}_3 = \hat{d}_4 = 0$.

The final obtained solution from Phase 1 of (MNCM) is presented in Table 2; basic variables are bolded.

Table 2. *Obtained solution from Phase 1 of (MNCM).*

2			
3	5	8	
	6	7	22

Since $x_{34} = 22 > 15$, this solution is not feasible and we should go to Phase 2.

Phase 2: The circle corresponding to the cell (3,4) is {(3,4), (2,4), (2,3), (3,3)} and $\Delta = 7$.

Updating variables in the circle yields $x_{34} = 7,\ x_{24} = 7,\ x_{23} = 1,\ x_{33} = 14$, which is feasible. The obtained BFS is:

Table 3. *Obtained solution from Phase 2 of (MNCM).*

2			
3		1	7
	6	14	

$x_{22} = 5,\ x_{34} = 15$. Other nonbasic variables are zero. Since $x_{33} = 14 = u_{33}$, we have degeneracy. The associated transportation cost is:

$$z = (2\times10) + (3\times15) + (5\times18) + (1\times12) + (7\times16) + (6\times16) + (14\times13) + (15\times14) = 767.$$

Example 2: Consider a balanced transportation problem with the same number of variables as the previous example. Values of c_{ij} $u_{ij} - l_{ij}$ s_i d_j are stated in Table 4.

Table 4. *Values of $c_{ij}, u_{ij} - l_{ij}, s'_i, d'_j$ for Example 2.*

	1		2		3		4		Supply
1	2	10	3	7	4	6	1	9	12
2	-1	5	1	9	3	8	2	7	18
3	5	6	4	7	6	10	3	11	15
Demands	5		20		10		10		

The final solution which is obtained from Phase 1 of (MNCM) is presented in Table 5; basic variables are bolded. Since $x_{34} = 9 < 11$, this solution is feasible and the algorithm stops at Phase 1.

Table 5. *Obtained solution from Example 2.*

5	**7**		
	9	8	**1**
	4	**2**	**9**

The associated transportation cost is:

$$z = (5\times 2)+(7\times 3)+(9\times 1)+(8\times 3)+(1\times 2)+(4\times 4)+(2\times 6)+(9\times 3)=96$$

3.2. Modified Least Cost Method (MLCM)

This method usually provides a better initial basic feasible solution than the North-West Corner method, since it takes into account the cost variables in the problem. We should modify the original algorithm in order to deal with bounds of variables. Again the algorithm has two phases.

Phase 1: The algorithm begins with $\hat{s}_i = s_i$, $\hat{d}_j = d_j$.

Step 1: Determine cell (i, j) with the smallest unit cost in the existing tableau. If this cell is not unique, choose one arbitrarily. Set $x_{ij} = \min\{\hat{s}_i, \hat{d}_j, u_{ij}\}$.

Step 2:

Case i: If $x_{ij} = \hat{s}_i \le \{\hat{d}_j, u_{ij}\}$, then the i-th row will be crossed out.

Case ii: If $x_{ij} = \hat{d}_j \le \{\hat{s}_i, u_{ij}\}$, then the j-th column will be crossed out.

Case iii: If $x_{ij} = u_{ij} < \{\hat{s}_i, \hat{d}_j\}$, then x_{ij} will be nonbasic at its upper bound and we should continue searching the smallest unit cost in row i and column j. At last row i or column j will be crossed out.

$\hat{s}_i \to \hat{s}_i - x_{ij}$, $\hat{d}_j \to \hat{d}_j - x_{ij}$. Go back to Step 1 with the new tableau in which a row or a column is less compare with the previous tableau.

Step 3: When exactly one row or column is left, all the remaining variables are basic. We should have $m+n-1$ basic variables in total.

Step 4: Suppose that the last remaining cell is (k, l). Set $x_{kl} = \min\{\hat{s}_k, \hat{d}_l\}$ $(\hat{s}_k = \hat{d}_l)$.

Remark 1: If in one iteration, a row and a column are both satisfied, i.e. $\hat{s}_i = \hat{d}_j = 0$, then only one of them will be crossed out. It is better to keep the row or column with the smaller c_{ij} s.

Remark 2: In case iii, updating $\hat{s}_i$ and $\hat{d}_j$ is repeated until one of them becomes zero.

Like the (MNCM), if in one iteration $\hat{s}_i = \hat{d}_j = 0$, $\hat{s}_i = u_{ij}$ or $\hat{d}_j = u_{ij}$, degeneracy occurs. When Phase 1 terminates, if $x_{kl} \le u_{kl}$, current solution is feasible, otherwise we should go to Phase 2 which is exactly the same as Phase 2 of (MNCM).

Now, we solve Example 1 by (MLCM).

Example 3: Consider the capacitated transportation problem of Example 1. Applying (MLCM), yields:

Iteration 1: $c_{14} = 8$ is the least unit cost. $x_{14} = \min\{2, 22, 11\} = 2$, $\hat{s}_1 = 0$,$\hat{d}_4 = 20$, cross out row 1.

Iteration 2: $c_{23} = 12$ is the least unit cost. $x_{23} = \min\{16, 15, 9\} = 9$, x_{23} becomes nonbasic at its upper bound and $\hat{s}_2 = 7$,$\hat{d}_3 = 6$. The next cell with the least unit cost in row 2 and column 3 is $c_{33} = 13$. $x_{33} = \min\{35, 6, 14\} = 6$, $\hat{s}_3 = 29$, $\hat{d}_3 = 0$. Cross out column 3.

Iteration 3: $c_{34} = 14$ is the least unit cost. $x_{34} = \min\{29, 20, 15\} = 15$, x_{34} becomes nonbasic at its upper bound, $\hat{s}_3 = 14$,$\hat{d}_4 = 5$. The next cell with the least unit cost in row 3 and column 4 is $c_{24} = 16$. $x_{24} = \min\{7, 5, 10\} = 5$, $\hat{s}_2 = 2$,$\hat{d}_4 = 0$. Cross out column 4.

Iteration 4: $c_{21} = 15$ is the least unit cost. $x_{21} = \min\{2, 5, 14\} = 2$, $\hat{s}_2 = 0$, $\hat{d}_1 = 3$ .Cross out row 2.

Iteration 5: Row 3 is the last row, x_{31} and x_{32} will be basic. Since $c_{32} < c_{31}$, we first determine. $x_{32} = \min\{14, 11, 12\} = 11$, $\hat{s}_3 = 3$,$\hat{d}_2 = 0$ and the last variable is $x_{31} = \min\{3, 3\} = 3$, $\hat{s}_3 = \hat{d}_1 = 0$.

Phase 1 terminates and we have $x_{31} = 3 < 8$, so the current solution is feasible and algorithm stops at Phase 1. The following table shows the result of this example. Basic variables are bolded.

Table 6. *Obtained solution of (MLCM).*

			2
2		9	5
3	11	6	15

The associated transportation cost is:

$z = (2 \times 8) + (2 \times 15) + (9 \times 12) + (5 \times 16) + (3 \times 17) + (11 \times 16) + (6 \times 13) + (15 \times 14) = 749$ Which is less than the cost computed by (MNCM).

3.3. Modified Vogel's Approximation Method (MVAM)

Vogel's Approximation Method generally yields an optimum or close to optimum solution. The only difference between (MVAM) and (MLCM) is in (MVAM) determining cell (i, j) with the smallest unit cost is done in a row or column with the largest penalty, not in the entire tableau. This penalty is the difference between two smallest c_{ij}'s in a row or a column.

Example 4: Consider the capacitated transportation problem of Example 1. We explain the first iteration of (MVAM). u_i and v_j stand for penalty of row i and column j, respectively.

Iteration 1: $u_1 = 2,\ u_2 = 3,\ u_3 = 1,\ v_1 = 5,\ v_2 = 4,\ v_3 = 1,\ v_4 = 6$. Column 4 has the largest penalty. The cell with the smallest unit cost in column 4 is (1,4). $x_{14} = \min\{2,22,8\} = 2$, $\hat{s}_1 = 0, \hat{d}_4 = 20$. Cross out row 1.

After completing Phase 1, we have the following BFS. This BFS is just like the BFS which was obtained from (MLCM).

Table 7. *Obtained solution of (MVAM).*

			2
2		9	5
3	11	6	15

4. Transportation Simplex for Bounded Variables

The general steps that are taken in simplex method are:

- Finding a starting basic feasible solution.
- Computing $z_j - c_j$ for each nonbasic variables.
- Determining the entering and the leaving variables.
- Updating the basis.

Step 1 was investigated in the previous section. Since for capacitated transportation some of the nonbasic variables may be at their upper bound, entering and leaving basis is a little different; but computing $\bar{c}_{ij} = c_{ij} - z_{ij}$ for nonbasic variables is done in a same way [1].

Determining entering variable

Suppose that R_1 is the set of indices of nonbasic variables at their lower bound and R_2 is the set of indices of nonbasic variables at their upper bound. For determining entering variable, compute

$$\Delta = \max\{\max_{(i,j)\in R_1} -\bar{c}_{ij} \ ,\ \max_{(i,j)\in R_2} \bar{c}_{ij}\}.$$

If $\Delta \le 0$, the current solution is optimal. For $\Delta > 0$ suppose that (k,l) is the index for which the maximum is achieved. If $(k,l) \in R_1$ then x_{kl} is increased from its current level of zero. If $(k,l) \in R_2$, then x_{kl} is decreased from its current level of u_{kl}.

Determining leaving variable

There exists a unique cycle starting from cell (k,l). All corners of this cycle are basic except for (k,l). If $(k,l) \in R_1$, we allocate a positive sign to the cell (k,l). This sign changes to negative for the adjacent cell and so on. If $(k,l) \in R_2$, sign of the cell (k,l) is negative and other signs change accordingly. Let T_1 be the set of indices of basic variables in the circle which have positive signs and T_2 be the set of indices of basic variables in the circle which have negative signs. Set

$\delta_1 = \min_{(i,j)\in T_1}\{u_{ij} - x_{ij}\}$, $\delta_2 = \min_{(i,j)\in T_2}\{x_{ij}\}$, and compute $\delta = \min\{\delta_1, \delta_2, u_{kl}\}$.

If $\delta = \delta_1$ the associated basic variable leaves the basis and it will be nonbasic at its upper bound.

If $\delta = \delta_2$ the associated basic variable leaves the basis and it will become zero.

If $\delta = u_{kl}$ the basis doesn't change and x_{kl} is still nonbasic; its bound changes from upper to lower or vice versa. Only the value of basic variables in the circle will change.

Updating the basis

After determining the leaving variable, we should update the variables in the circle by δ according to sign of the cells, i.e. $x_{ij} \to x_{ij} + \delta$ for (i,j) with positive sign and $x_{ij} \to x_{ij} - \delta$ for (i,j) with negative sign.

When we update the transportation tableau, again $\bar{c}_{ij}$'s are calculated. This process is done until $\Delta \le 0$ and we get to optimality.

Example 5: Consider capacitated transportation of Example 1. We apply simplex algorithm starting from the solution of (MVAM). There are two methods for computing $\bar{c}_{ij}$ which is not included here (see [1]).

Iteration 1: As you can see in Table 7, we have

$R_1 = \{(1,1),\ (1,2),\ (1,3),\ (2,2)\}$, $R_2 = \{(2,3),\ (3,4)\}$.

By computing $\bar{c}_{ij}$ for nonbasic variables, we have $\bar{c}_{11} = 3,\ \bar{c}_{12} = 6,\ \bar{c}_{13} = 10,\ \bar{c}_{22} = 4,\ \bar{c}_{23} = 1,\ \bar{c}_{34} = -4.$

So $\Delta = \max\{-3,-6,-10,-4,1,-4\} = 1$,which is achieved in cell $(2,3)$. Hence x_{23} is the entering variable and the corresponding cycle is $\{(2,3),\ (2,1),\ (3,1),\ (3,3)\}$.

Since sign of the cell $(2,3)$ is negative, we have $T_1=\{(2,1),(3,3)\}$ and $T_2=\{(3,1)\}$; also $\delta_1=\min\{14-2,14-6\}=8$, $\delta_2=3$. Thus $\delta=\min\{8,3,9\}=3$ which is achieved in cell (3,1). x_{31} is the leaving variable and the new values of the variables in the cycle are

$$x_{23}=9-3=6 \ , \ x_{21}=2+3=5 \ , \ x_{31}=3-3=0 \ , \ x_{33}=6+3=9.$$

The new solution is

Table 8. *Optimal solution of Example 1.*

			2
5		6	5
	11	9	15

Iteration 2: we have $R_1=\{(1,1),\ (1,2),\ (1,3),\ (2,2),\ (3,1)\}$, $R_2=\{(3,4)\}$.
$\bar{c}_{11}=3,\ \bar{c}_{12}=5,\ \bar{c}_{13}=9,\ \bar{c}_{22}=3,\ \bar{c}_{31}=1,\ \bar{c}_{34}=-3,$ so $\Delta=\max\{-3,-5,-9,-3,-1,-3\}=-1<0$.

The algorithm stops and the current solution is optimal. The optimal transportation cost is:

$$z=(2\times8)+(5\times15)+(6\times12)+(5\times16)+(11\times16)+(9\times13)+(15\times14)=746.$$

5. Comparison with a Least Cost Method

At first, we state a summary of the classic least cost method for finding an initial feasible solution of capacitated transportation problem [9], then we solve an example by this method and our proposed algorithm.

Finding an initial basic feasible solution

- The cell with the minimum cost in the table is selected and assigned the maximum value possible. If this assignment fully satisfies either the row's supply or the column's demand, then the variable is called a basic variable. Otherwise, if the assigned value is limited with the upper bound value of the cell, then it is called a bounded variable. This step is repeated until there exists no possible assignment.
- The table is checked to determine if all of the demands and supplies are fully satisfied. If this is not the case, then a row (row 0) and a column (column 0) are added to the transportation table. Suppose that cell (k,l) is the last cell in Step 1. A new table is formed with $c_{k0}=c_{0l}=1$ and $c_{ij}=0$ for other cells. The sum of the artificial variables is minimized by simplex method. This step is repeated until artificial variables leave the basis and a feasible solution for the original problem is achieved.

Example 6: Consider the following capacitated transportation problem.

Table 9. *Values of* $c_{ij}, u_{ij}-l_{ij}, s'_i, d'_j$.

	1		2		3		4		Supply
1	10	12	5	13	6	5	7	25	25
2	9	7	3	4	4	25	8	18	50
3	8	14	2	20	7	10	6	9	25
Demands	15		20		30		35		

Applied to Table 9, the first step of the mentioned method yields the following assignments; in order

$x_{32}=20$ (basic), $x_{23}=25$ (bounded), $x_{13}=5$ (basic), $x_{34}=5$ (basic), $x_{14}=20$ (basic), $x_{24}=10$ (basic), $x_{21}=7$ (bounded).

Since the second row and first column still have 8 units unassigned, the solution is not feasible.

Iteration 1: The sum of artificial variables $x_{20}+x_{01}$ should be minimized over the following table.

Table 10. *Transportation table of Iteration 1.*

0	8	**8**			8		8		8	$u_0=0$
		8								
	0				0		0		0	
			1							
	8		12		13	**5**	5	**20**	20	$u_1=1$
	0		0		0		0		0	
8	8		7		4		25	**10**	18	$u_2=1$
	1		0		0		0		0	
	8		14	**20**	20		10	**5**	9	$u_3=1$
	0		0		0		0		0	
$v_0=0$		$v_1=1$		$v_2=-1$		$v_3=-1$		$v_4=-1$		

Basic variables are bolded, $u_{ij}-l_{ij}$ s are in the right corner of the cells. All c_{ij} s are zero except for c_{01} and c_{20} which are equal to 1. u_i and v_j are calculated in the right and below of the table. Computing $\bar{c}_{ij}$ by MODI method yields $\bar{c}_{02}=\bar{c}_{03}=\bar{c}_{04}=1$, $\bar{c}_{10}=-1$, $\bar{c}_{11}=-2$, $\bar{c}_{12}=0$, $\bar{c}_{21}=-2$, $\bar{c}_{22}=\bar{c}_{23}=0$, $\bar{c}_{30}=-1$, $\bar{c}_{31}=-2$, $\bar{c}_{32}=0$.

$\Delta=2$, x_{31} is the entering variable and the corresponding cycle is $\{(3,1),(3,4),(2,4),(2,0),(0,0),(0,1)\}$. We have $\delta=\min\{5,8\}=5$ and x_{34} should leave the basis. After updating the variables in the cycle, we have the following table.

Table 11. *Transportation table of Iteration 2.*

5	8	**3**	8		8		8		8	$u_0 = 0$
	0		1		0		0		0	
	8		12		13	**5**	5	**20**	20	$u_1 = 1$
	0		0		0		0		0	
3	8		7		4		25	**15**	18	$u_2 = 1$
	1		0		0		0		0	
	8	**5**	14	**20**	20		10		9	$u_3 = -1$
	0		0		0		0		0	
$v_0 = 0$		$v_1 = 1$		$v_2 = 1$		$v_3 = -1$		$v_4 = -1$		

Iteration 2:

$\bar{c}_{02} = -1$, $\bar{c}_{03} = \bar{c}_{04} = 1$, $\bar{c}_{10} = -1$, $\bar{c}_{11} = \bar{c}_{12} = \bar{c}_{21} = \bar{c}_{22} = -2$, $\bar{c}_{23} = 0$, $\bar{c}_{30} = 1$, $\bar{c}_{33} = \bar{c}_{34} = 2$

$\Delta = 2$. x_{11} is the entering variable and the corresponding cycle is $\{(1,1),(1,4),(2,4),(2,0),(0,0),(0,1)\}$. We have $\delta = \min\{3,8,20\} = 3$. After updating variables in the cycle, the new value of artificial variables x_{01} and x_{20} will be zero and we get to a feasible solution of the original problem which can be seen in Table 12. $x_{21} = 7$ and $x_{23} = 25$ are bounded variables.

Table 12. *Feasible solution of Least Cost Method.*

3		5	17
7		25	18
5	20		

The associated transportation cost is:

$$z = (3\times10) + (5\times6) + (17\times7) + (7\times9) + (25\times4) + (18\times8) + (5\times8) + (20\times2) = 566.$$

Finding initial basic feasible solution by (MLCM)

Phase 1 of (MLCM) method generates the following solution:

Table 13. *Obtained solution of phase 1 of (MLCM) for example 6.*

		5	20
15		25	10
	20		5

Since $x_{21} = 15 > 7$, Phase 2 is performed. The circle corresponding to the cell (2,1) is {(2,1), (2,4), (1,4), (1,1)} and $\Delta = 8$. Updating variables in the circle yields $x_{21} = 7,\ \ x_{24} = 18,\ \ x_{14} = 12,\ \ x_{11} = 8$ which is feasible. The obtained BFS is:

Table 14. *Obtained solution from Phase 2 of (MLCM).*

8		5	12
7		25	18
	20		5

The associated transportation cost is:

$$z = (8\times10) + (5\times6) + (12\times7) + (7\times9) + (25\times4) + (18\times8) + (20\times2) + (5\times6) = 571.$$

As it is evident from the above example, our proposed algorithm generates a feasible solution for capacitated transportation problem with less computations and without changing size of the problem. The mentioned Least Cost Method needs some simplex iterations to get to a starting feasible solution; but our proposed method obtains a feasible solution just by one or some updating steps in Phase 2.

6. Conclusion

Capacitated transportation problem is a special case of bounded linear programming problems. It has many practical applications in various areas including inventory control, employment scheduling, telecommunication networks, and personnel assignment. In this research, we used three methods: Modified Northwest Corner Method (MNCM), Modified Least Cost Method (MLCM), and Modified Vogel's Approximation Method (MVAM) to find an initial BFS of this problem. For obtaining optimal solution of the problem, the simplex algorithm with detailed explanation used. All steps of our method operate directly on the table and we don't need to add any row or column to the tableau.

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