Journal of Applied Research on Industrial Engineering
www.journal-aprie.com

On Solutions of Fuzzy Multi-Objective Programming Problems through Weighted Coefficients in Two-Phase Approach

Hamiden Abdelwahed Khalifa *

Department of Operations Research, Institute of Statistical Studies and Research, Cairo University, Giza, Egypt.

PAPER INFO

Chronicle:
Received: 06 March 2018
Accepted: 30 August 2018

Keywords:
Multi-Objective Programming Problems.
Fuzzy Parameters.
Fuzzy Numbers.
$\alpha-$ Efficient Solution.
Two-Phase Approach.
Compromise Index.

ABSTRACT

In this paper, a Fuzzy Multi-Objective Linear Programming (FMOLP) problem having both objective functions and constraints fuzzy parameters is introduced. Theses fuzzy parameters are characterized by trapezoidal fuzzy numbers. The FMOLP problem is converted into the corresponding deterministic MOLP problem through the use of intervals arithmetic operations. Then, a two-phase approach having equal weighted coefficients is proposed to generate an $\alpha-$ efficient solution for the MOLP problem. The major advantage of the new model is that the proposed approach as long as the weighted coefficients not necessarily equal and generates an efficient solution. A numerical example is given to clarify the obtained results.

1. Introduction

In general, there is no single optimal solution in multi-objective problems, but rather there is a set of inferior (pareto optimal) solutions from which the Decision Maker (DM) must select the most preferred or best compromise solution as the one to implement. Multi-objective analysis assumes that the objectives are generally in conflict. When a Multi-Objective Linear Programming (MOLP) problem be modeled, how to calculate the exact values of the coefficients is a probabilistic task.

As known, the fuzzy set theory was introduced by Zadeh [20] to deal with fuzziness. Up to now, the fuzzy set theory has been applied to broad fields. Fuzzy set theory has to be set up using data which is approximately known. Fuzzy numerical data can be represented by means of fuzzy subsets of the real line, known as fuzzy numbers. For the fuzzy set theory development, we may referee to the papers of Kaufmann and Gupta [9], and Dubois [2]; they extended the use of algebraic operations of real numbers to fuzzy numbers by the use a fuzzy faction principle. Fuzzy linear constraints with fuzzy numbers were studied by Dubois [2]. Normally, the coefficients are either given by a DM subjectively or by statistical inference from historical data. Therefore, to reflect this uncertainty, we need to construct a model with inexact coefficients. Bitran [1] and Steuer [15] proposed different algorithms to solve a MOLP problem in which the cost coefficients are interval-valued. They applied the vector-maximum theory introduced by Philip [12] to find the efficient extreme points. The authors of [18] studied MOLP problem with

* Corresponding author
E-mail address: hamiden_2008@yahoo.com
DOI: 10.22105/jarie.2018.130777.1037

fuzzy-numbered cost coefficients. They based on the membership functions to transform the considered problem into the parametrically interval-valued MOLP problem. Shaocheng [16] developed a solution procedure to cope with interval-valued linear programming problems to obtain an interval-valued optimal solution. The work [18] proposed a solution procedure for a MOLP with interval-valued coefficients.

Veeramani et al. [17] studied the fuzzy MOLP problem with fuzzy technological coefficients and resources. They solved the problem based on the proposed method introduced by Gasimov and Yenimez [7]. Singh and Yadav [10] reduced the Fuzzy MOLP problem to the corresponding ordinary one using the ranking function and hence solved it using the fuzzy programming technique. Hamadameen [8] proposed a technique for solving fuzzy MOLP problem in which the objective functions coefficients are triangular fuzzy numbers. Under fuzzy environment, Grag [3] presented an alternative method for computing the various arithmetic operations of a system using the sigmoidal number. Based on optimistic and pessimistic view point, Rani et al. [13] investigated an algorithm for solving multi-objective optimization problem. Through the use of concept of the distribution and complementary distribution functions, Garg [4] studied the basic arithmetic operations for two generalized positive parabolic fuzzy numbers. Depending on the definition of different types of fuzzy numbers, namely gamma, normal, Cauchy, and triangular for uncertainty, Garg [6] investigated the performance and sensitivity analysis of the systems at different levels of confidence. Also, the performance towards the data provided by the decision makers. Under uncertainty, vague, and imprecise of data, Garg [5] suggested an alternative approach for solving multi-objective reliability optimization problem.

In this paper, a two- phase approach having equal weighted coefficients is proposed to generate an $\alpha-$ efficient solution for multi-objective programming problem with fuzzy parameters in the objective functions and constraints. We show that the proposed approach is as long as the weighted coefficients not necessary equal, and generate an efficient solution. The remainder of the paper is organized as the following sections: In Section 2, some preliminary needs in the paper are presented. In Section 3, multi-objective linear programming problem with fuzzy parameters, i.e. both the objective functions and constraints is introduced as the specific definition and properties. In Section 4, a two-phase approach for solving the problem is given. In Section 5, an illustrative numerical example is given to clarify the obtained results. Finally, some concluding remarks are reported in Section 6.

2. Preliminaries

In order to discuss our problem conveniently, we shall state some necessary results on interval arithmetic and fuzzy numbers (see [9, 11]).

Let $A=\left\{\left[a^L, a^U\right]: a^L, a^U \in R=(-\infty,\infty), a^L \le a^U\right\}$ denote the set of all closed interval numbers on R, where a^L is the left limit and a^U is the right limit of A. Also, the interval A may be denoted by its center and width as

$A=\left\{\left\langle a^C, a^W\right\rangle : a^C, a^W \in R, a^C - a^W \le a^C + a^W\right\}$, where a^C is the center and a^W is the width of A which may be calculated as $a^C=\frac{1}{2}\left(a^U+a^L\right)$, and $a^W=\frac{1}{2}\left(a^U-a^L\right)$.

Definition 1. Let $*\in\{+, -, \times, \div\}$ be a binary operation on R. If A and B are closed intervals, then

$A*B=\{a*b: a\in A, b\in B\}$, and $A(+)B=\begin{cases}\left[a^L+b^L,\ a^U+b^U\right], \\ \left\langle a^C+b^C,\ a^W+b^W\right\rangle\end{cases}$ $kA=\begin{cases}\left[k\,a^L, k\,a^U\right], \\ \left\langle ka^C,\ ka^W\right\rangle\end{cases}$ where $k\in R.$

The order relation $(\leq^{LU})$ between $A=\left[a^L, a^U\right]$, and $B=\left[b^L, b^U\right]$ is defined as

$$A\leq^{LU} B \text{ if and only if } a^L\leq b^L, \text{ and } a^U\leq b^U,$$

$$A<^{LU} B \text{ if and only if } A\leq^{LU} B, \text{ and } A\neq B,$$

Also, the order relation $(\leq^{CW})$ between $A=\left[a^C, a^W\right]$, $B=\left[b^C, b^W\right]$ is defined as

$$A\leq^{CW} B \text{ if and only if } a^C\leq b^C, \text{ and } a^W\leq b^W,$$

$$A<^{CW} B \text{ if and only if } A\leq^{CW}, \text{ and } A\neq B.$$

Proposition 1. If $A\leq^{LU} B$, and $B\leq^{CW} A$ hold, then $A=B.$

Definition 2. Let R be the set of real numbers, the fuzzy number $\tilde{a}$ is a mapping $\mu_{\tilde{a}}: R\to[0, 1]$, with the following properties:

$\mu_{\tilde{a}}(x)$ is an upper semi-continuous membership function; $\tilde{a}$ is a convex set, i. e. $\mu_{\tilde{a}}(\lambda x^1+(1-\lambda)x^2)\geq \min\{\mu_{\tilde{a}}(x^1), \mu_{\tilde{a}}(x^2)\}$, for all $x^1, x^2\in R, 0\leq\lambda\leq 1$; (c) $\tilde{a}$ is normal, i. e., $\exists x_0\in R$ for which $\mu_{\tilde{a}}(x)=1$; Supp $(\tilde{a})=\{x: \mu_{\tilde{a}}(x)>0\}$ is the support of a fuzzy set $\tilde{a}$.

Let $F_0(R)$ denote the set of all compact fuzzy numbers on R that is for any $g\in F_0(R), g$ satisfies the following:

$\exists x\in R: g(x)=1$; for any $0<\alpha\leq 1$, $g_\alpha=[g_\alpha^L, g_\alpha^U]$ is a closed interval number on $R.$ It is noted that $R\subset I(R)\subset F_0(R).$

Definition 3. The $\alpha-$ level set of the fuzzy number $\tilde{a}$ is defined as the ordinary set $L_\alpha(\tilde{a})$ for which the degree of their membership function exceeds the level α :

$$L_\alpha(\tilde{a})=\{a: \mu_{\tilde{a}}(a)\geq\alpha\}.$$

Definition 4. A trapezoidal fuzzy number can be represented completely by a quadruplet $\tilde{A}=(a_1, a_2, a_3, a_4)$ and its interval of confidence at level α is defined by:

$$\tilde{A}_\alpha=[(a_2-a_1)\alpha+a_1, -(a_4-a_3)\alpha+a_4], \forall\, 0<\alpha\leq 1.$$

3. Problem Formulation and Solution Concepts

Consider the following fuzzy multi-objective linear programming (F-MOLP) problem

(F-MOLP)

$$\max \tilde{Z}_k(x,\tilde{c}) = (\tilde{c}_1 x, \tilde{c}_2 x, \tilde{c}_3 x, \ldots, \tilde{c}_l x)^T, k = 1,2,\ldots,l$$
$$\min \tilde{W}_s(x,\tilde{d}) = (\tilde{d}_1 x, \tilde{d}_2 x, \tilde{d}_3 x, \ldots, \tilde{d}_r x)^T, s = 1,2,\ldots,r$$

Subject to

$$x \in \tilde{X} = \{x \in R^n : \tilde{A}x \le \tilde{b}, x \ge 0\}$$

where $\tilde{c}_i \in R^n, \forall i, \tilde{d}_j \in R^n, \forall j, \tilde{A} \in R^{m\times n}$, and $\tilde{b} \in R^m$ are fuzzy parameters.

Definition 5. (Fuzzy efficient solution). A point $x^* \in \tilde{X}$ is said to be fuzzy efficient solution to the F-MOLP problem if and only if there does not exist another $x \in \tilde{X}$, such that: $Z_k(x,\tilde{c}) \ge Z_k(x^*,\tilde{c}^*)$, and $W_s(x,\tilde{d}) \le W_s(x^*,\tilde{d}^*)$, and $\tilde{Z}_k(x,\tilde{c}) \ne \tilde{Z}_k(x^*,\tilde{c}^*)$ or $W_s(x,\tilde{d}) \ne W_s(x^*,\tilde{d}^*)$.

Assuming that these fuzzy parameters $\tilde{c}_i\,(i = 1,2,\ldots,l)$, $\tilde{d}_j\,(j = 1,2,\ldots,r)$, $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$, and $\tilde{b}_i \in R^m$ are characterized by fuzzy numbers ([2]); let the corresponding membership functions be $\mu_{\tilde{c}_i}(c_i), i = 1,2,\ldots,n$; $\mu_{\tilde{d}_j}(d_j), j = 1,2,\ldots,n$; $\mu_{\tilde{a}_{ij}}(\tilde{a}_{ij})$, and $\mu_{\tilde{b}_i}(b_i)$. We introduce the α-level set of the fuzzy numbers $\tilde{c}$, $\tilde{d}$, $\tilde{A}$ and $\tilde{b}$ defined as the ordinary set $(\tilde{c},\tilde{d},\tilde{A},\tilde{b})_\alpha$ in which the degree of their membership functions exceeds level α.

$$(\tilde{c},\tilde{d},\tilde{A},\tilde{b})_\alpha = \{(c,d,A,b) : \mu_{\tilde{c}_j}(c_j) \ge \alpha, j = 1,2,\ldots,n; \mu_{\tilde{d}_j}(d_j) \ge \alpha, j = 1,2,\ldots,n; \mu_{\tilde{A}}(A) \ge \alpha, \mu_{\tilde{b}_i}(b_i) \ge \alpha\}.$$

For a certain degree of α, the (F-MOLP) problem can be written as in the following non fuzzy form ([14]) as

($\alpha-$ MOLP)

$$\max Z_k(x,c) = (c_1 x, c_2 x, \ldots, c_l x)^T, k = 1,2,\ldots,l$$
$$\min W_s(x,d)s = (d_1 x, d_2 x, \ldots, d_r x)^T, s = 1,2,\ldots,r$$

Subject to

$$x \in X = \{x \in R : Ax \le b, x \ge 0\},$$
$$(c,d,A,b) \in (\tilde{c},\tilde{d},\tilde{A},\tilde{b})_\alpha,\ \alpha \in (0,1].$$

Definition 6. (α-efficient solution). A point $x^* \in X$ is said to be an α-efficient solution to the α-MOLP problem if and only if there does not exist another $x \in X$, $(c,d,A,b) \in (\tilde{c},\tilde{d},\tilde{A},\tilde{b})_\alpha$ such that $Z(x,c) \ge Z(x^*,c^*), W(x,d) \le W(x^*,d^*)$ and $Z(x,c) \ne Z(x^*,c^*)$ or $W(x,d) \ne W(x^*,d^*)$, where the corresponding values of parameters (c^*,d^*,A^*,b^*) are called $\alpha-$level optimal parameters.

The intervals of confidence corresponding to the $\tilde{c}, \tilde{d}, \tilde{A}, \tilde{b}$ can be denoted as follows:

$\left[c^L, c^U\right], \left[d^L, d^U\right], \left[A^L, A^U\right]$, and $\left[b^L, b^U\right]$. So, the ($\alpha-$ MOLP) problem can be rewritten as in the following form:

(IV-MOLP)

$$\max Z_k(x,c) = \left(\left[(c_1)^L, (c_1)^U\right]x, \left[(c_2)^L, (c_2)^U\right]x, \ldots, \left[(c_l)^L, (c_l)^U\right]x\right)^T, \ k = 1,2,\ldots,l$$

$$\min W_s(x,d)\, s = \left(\left[(d_1)^L, (d_2)^U\right]x, \left[(d_2)^L, (d_2)^U\right]x, \ldots, \left[(d_r)^L, (d_r)^U\right]x\right)^T, s = 1,2,\ldots,r$$

Subject to

$$x \in X = \left\{x \in R : \left[A^L, A^U\right]x \le \left[b^L, b^U\right], x \ge 0\right\}.$$

Definition 7. $x \in X$ is said to be an efficient solution of (IV-MOLP) if and only if there is no $x^\circ \in X$ which satisfies $Z_k(x, c^\circ) <^{LR} Z_k(x^\circ, c^\circ)$ or $Z_k(x, c^\circ) <^{CW} Z_k(x^\circ, c^\circ), k = 1, 2, \ldots, l$, and $W_s(x^\circ, d^\circ) <^{LR} W_s(x, d^\circ)$ or $W_s(x^\circ, d^\circ) <^{CW} W_s(x, d^\circ), s = 1, 2, \ldots, r.$

The (IV-MOLP) problem may be transformed into the following MOLP problem as follows:

(MOLP)

$$\max \left(Z_k(x,c)\right)^L = \left((c_1)^L x, (c_2)^L x, \ldots, (c_l)^L x\right)^T$$

$$\max \left(Z_k(x,c)\right)^C = \left((c_1)^C x, (c_2)^C x, \ldots, (c_l)^C x\right)^T$$

$$\min \left(W_s(x,d)\right)^R = \left((d_1)^R x, (d_2)^R x, \ldots, (d_l)^R x\right)^T$$

$$\min \left(W_s(x,d)\right)^C = \left((d_1)^C x, (d_2)^C x, \ldots, (d_l)^C x\right)^T$$

Subject to $$x \in X = \left\{x \in R^n : A^L x \le b^L,\ A^U x \le b^U, x \ge 0\right\}.$$

Definition 8. $x \in X$ is called an efficient solution of MOLP problem if and only if there is no $x^\circ \in X$ which satisfies $Z_k(x, c^\circ) <^{LC} Z_k(x^\circ, c^\circ), k = 1, 2, \ldots, l$, and $W_s(x^\circ, d^\circ) <^{RC} W_s(x, d^\circ), s = 1, \ldots, r.$

Definition 9. For any two feasible solutions x and y of the MOLP problem, x is said to be more efficient than y if

$$\left\{\begin{matrix} \left(Z_k(x,c)\right)^L \ge \left(Z_k(y,c)\right)^L, \left(Z_k(x,c)\right)^C \ge \left(Z_k(y,c)\right)^C, and \left(W_s(x,d)\right)^R \le \left(W_s(y,d)\right)^R, \\ \left(W_s(x,d)\right)^C \le \left(W_s(y,d)\right)^C; \forall\, k, s \end{matrix}\right\}$$, and

$$\left\{ \begin{array}{l} (Z_j(x,c))^L > (Z_j(y,c))^L,\ (Z_j(x,c))^C > (Z_j(y,c))^C,\ for\, some\ j, or\, (W_t(x,d))^R < (W_t(y,d))^R, \\ (W_t(x,d))^C < (W_t(y,d))^C\ for\, some\, t \end{array} \right\}.$$

This is denoted by $x \succ y$.

4. Two-Phase Approach

Based on the min operator model proposed by Zimmermann [20] for the linear programming having several objectives; the MOLP problem can be viewed as

$$\max \delta$$

Subject to

$$\begin{array}{l} \delta \le \dfrac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, k = 1,2.,\ldots,l \\ \delta \le \dfrac{(Z_k(x,c))^C - (Z_k(x,c))^{C^*}}{(Z_k(x,c))^{C^{**}} - (Z_k(x,c))^{C^*}}, k = 1,2,\ldots,l \\ \delta \le \dfrac{(W_s(x,d))^{R^{**}} - (W_s(x,d))^R}{(W_s(x,d))^{R^{**}} - (W_s(x,d))^{R^*}}, s = 1,2,\ldots,r \\ \delta \le \dfrac{(W_s(x,d))^{C^{**}} - (W_s(x,d))^C}{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C^*}}\ s = 1,2,\ldots,r \\ A^L x \le b^L,\ A^U x \le b^U, x \ge 0. \end{array} \quad (1)$$

Where $(Z_k(x,c))^{L^*}, (Z_k(x,c))^{C^*}, (W_s(x,d))^{R^*}, (W_s(x,d))^{C^*}$, and $(Z_k(x,c))^{L^{**}}$, $(Z_k(x,c))^{C^{**}}$, $(W_s(x,d))^{R^{**}}, (W_s(x,d))^{C^{**}}$ are the ideal and anti-ideal solutions for the (MOLP), respectively.

Assuming that the membership function for each objective of the MOLP problem is equally important, so the model (1) is converted into the following so-called average model as

$$\max \bar{\delta} = \frac{1}{2(l+r)} \left(\sum_{k=1}^{l} \delta_k^{'} + \sum_{k=1}^{l} \delta_k^{''} + \sum_{s=1}^{r} \delta_s^{'} + \sum_{s=1}^{r} \delta_s^{''} \right)$$

Subject to

$$\begin{array}{l} \delta_k^{'} \le \dfrac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, k = 1,2,\ldots,l \\ \delta_k^{''} \le \dfrac{(Z_k(x,c))^C - (Z_k(x,c))^{C^*}}{(Z_k(x,c))^{C^{**}} - (Z_k(x,c))^{C^*}}, k = 1,2,\ldots,l \\ \delta_s^{1} \le \dfrac{(W_s(x,d))^{R^{**}} - (W_s(x,d))^R}{(W_s(x,d))^{R^{**}} - (W_s(x,d))^{R^*}}, s = 1,2,\ldots,r \end{array} \quad (2)$$

$$\delta_s^2 \le \frac{(W_s(x,d))^C - (W_s(x,d))^{C^*}}{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C^*}}\ s = 1,2,\ldots,r$$

$$A^L x \le b^L,\ A^U x \le b^U, x \ge 0.$$

The combination of the min operator model (1) with the average operator model (2) results in the two-phase approach. It is easily to see from the following model:

$$\max \bar{\delta} = \frac{1}{2(l+r)}\left(\sum_{k=1}^{l} \delta_k^{'} + \sum_{k=1}^{l} \delta_k^{''} + \sum_{s=1}^{r} \delta_s^1 + \sum_{s=1}^{r} \delta_s^2\right)$$

Subject to

$$\begin{aligned}
&\delta^{\bullet} \le \delta_k^{'} \le \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, k = 1,2,\ldots,l \\
&\delta^{\bullet} \le \delta_k^{''} \le \frac{(Z_k(x,c))^C - (Z_k(x,c))^{C^*}}{(Z_k(x,c))^{C^{**}} - (Z_k(x,c))^{C^*}}, k = 1,2,\ldots,l \\
&\delta^{\bullet} \le \delta_s^1 \le \frac{(W_s(x,d))^R - (W_s(x,d))^{R^*}}{(W_s(x,d))^{R^{**}} - (W_s(x,d))^{R^*}}, s = 1,2,\ldots,r \\
&\delta^{\bullet} \le \delta_s^2 \le \frac{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C}}{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C^*}}\ s = 1,2,\ldots,r \\
&A^L x \le b^L,\ A^U x \le b^U, x \ge 0.
\end{aligned} \tag{3}$$

A sufficient condition to find the efficient solution of the MOLP problem by the two-phase approach is to have positive weighted coefficients. The following theorem proves this condition. Before apply the theorem, consider the following model:

$$\max \bar{\delta} = \sum_{i=1}^{l+r} \xi_i\, \delta_i$$

Subject to

$$\begin{aligned}
&\delta^{\bullet} \le \delta_k^{'} \le \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, k = 1,2,\ldots,l \\
&\delta^{\bullet} \le \delta_k^{''} \le \frac{(Z_k(x,c))^C - (Z_k(x,c))^{C^*}}{(Z_k(x,c))^{C^{**}} - (Z_k(x,c))^{C^*}}, k = 1,2,\ldots,l \\
&\delta^{\bullet} \le \delta_{l+s}^1 \le \frac{(W_s(x,d))^R - (W_s(x,d))^{R^*}}{(W_s(x,d))^{R^{**}} - (W_s(x,d))^{R^*}}, s = 1,2,\ldots,r \\
&\delta^{\bullet} \le \delta_{l+s}^2 \le \frac{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C}}{(W_s(x,d))^{C^{**}} - (W_s(x,d))^{C^*}}\ s = 1,2,\ldots,r \\
&A^L x \le b^L,\ A^U x \le b^U, x \ge 0.
\end{aligned} \tag{4}$$

Theorem 1. If (y, δ^y) is an optimal solution of the problem (4), then y is an efficient solution for the MOLP problem.

Proof. (The proof will be in contra positive direction)

Suppose that there is a feasible solution (x,δ^x) such that $x \succ y$. That is

$$\left\{\begin{array}{l}(Z_k(x,c))^L \geq (Z_k(y,c))^L,\ (Z_k(x,c))^C \geq (Z_k(y,c))^C, and\,(W_s(x,d))^R \leq (W_s(y,d))^R, \\ (W_s(x,d))^C \leq (W_s(y,d))^C; \forall k,s\end{array}\right\}, \text{ and}$$

$$\left\{\begin{array}{l}(Z_j(x,c))^L > (Z_j(y,c))^L,\ (Z_j(x,c))^C > (Z_j(y,c))^C,\ for\,some\ j, or\,(W_t(x,d))^R < (W_t(y,d))^R, \\ (W_t(x,d))^C < (W_t(y,d))^C\ for\,some\,t\end{array}\right\}$$

There are two cases:

Case 1: If $(Z_j(x,c))^L > (Z_j(y,c))^L$, $(Z_j(x,c))^C > (Z_j(y,c))^C$ for some j, then

$$\delta^{\bullet} \leq \delta_k^y \leq \frac{(Z_k(y,c))^L - (Z_k(y,c))^{L^*}}{(Z_k(y,c))^{L^{**}} - (Z_k(y,c))^{L^*}} \leq \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}; \forall k,$$

$$\delta^{\bullet} \leq \delta_k^y \leq \frac{(Z_k(y,c))^C - (Z_k(y,c))^{C^*}}{(Z_k(y,c))^{C^{**}} - (Z_k(y,c))^{C^*}} \leq \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}; \forall k, \text{ and}$$

$$\delta^{\bullet} \leq \delta_j^y \leq \frac{(Z_k(y,c))^L - (Z_k(y,c))^{L^*}}{(Z_k(y,c))^{L^{**}} - (Z_k(y,c))^{L^*}} < \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, \text{ for some } j$$

$$\delta^{\bullet} \leq \delta_j^y \leq \frac{(Z_k(y,c))^C - (Z_k(y,c))^{C^*}}{(Z_k(y,c))^{C^{**}} - (Z_k(y,c))^{C^*}} < \frac{(Z_k(x,c))^L - (Z_k(x,c))^{L^*}}{(Z_k(x,c))^{L^{**}} - (Z_k(x,c))^{L^*}}, \text{ for some } j$$

Since, $\delta_j^y \leq \dfrac{(Z_j(y,c))^L - (Z_j(y,c))^{L^*}}{(Z_j(y,c))^{L^{**}} - (Z_j(y,c))^{L^*}} \leq \delta_j^x = \dfrac{(Z_j(x,c))^L - (Z_j(x,c))^{L^*}}{(Z_j(x,c))^{L^{**}} - (Z_j(x,c))^{L^*}}$, and

$$\delta_j^y \leq \frac{(Z_j(y,c))^C - (Z_j(y,c))^{C^*}}{(Z_j(y,c))^{C^{**}} - (Z_j(y,c))^{C^*}} \leq \delta_j^x = \frac{(Z_j(x,c))^C - (Z_j(x,c))^{C^*}}{(Z_j(x,c))^{C^{**}} - (Z_j(x,c))^{C^*}}.$$

The objective value is

$$\sum_{i=1}^{2(l+r)} \xi_i\,\delta_i = \sum_{i=1, i\neq j}^{2(l+r)} \left(\xi_i\,\delta_i^y + \xi_j\,\delta_j^y\right) < \sum_{i=1, i\neq j}^{2(l+r)} \left(\xi_i\,\delta_i^x + \xi_j\,\delta_j^x\right).$$

Thus (y,δ^y) is not optimal solution to problem (4) contradicts the assumption.

Case 2: If $(W_t(x,d))^R < (W_t(y,d))^R$, $(W_t(x,d))^C < (W_t(y,d))^C$ for some t, the proof is similar to the case 1.

Remark 1. It is noted that the two-phase approach will fail to find an efficient solution in general for some of ξ_i's are zero.

5. Numerical Example

Consider the following (F-MOLP) problem

$$\begin{aligned} \max Z_1 &= (1,2,3,4)x_1 + (3,4,5,6)x_2 + (5,6,7,8)x_3 + (0,1,2,3)x_4 \\ Z_2 &= (2,3,4,5)x_1 + (0,1,2,3)x_2 + (1,2,3,4)x_3 + (9,10,11,12)x_4 \\ Z_3 &= ((8,9,10,11)x_1 + (2,3,4,5)x_2 + (0,1,2,3)x_3 + (1,2,3,4)x_4 \\ \min W_1 &= (0.5,1,1.5,2)x_1 + (1,2,3,4)x_2 + (0,0.1,0.3,0.4)x_3 + (2,3,4,5)x_4 \\ W_2 &= (0.1,0.3,0.5,0.7)x_1 + (0,1,2,3)x_2 + (0.3,0.5,0.7,0.9)x_3 + (1,2,3,4)x_4 \end{aligned} \tag{5}$$

Subject to

$$(2,3,4,5)x_1 + (0.5,1,1.5,2)x_2 + (0.5,1,1.5,2)x_3 + (6.5,7,7.5,8)x_4 = (50,100,150,200),$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

For $\alpha = 0.5$, problem (5) becomes

$$\begin{aligned} \max Z_1 &= [1.5,3.5]x_1 + [3.5,5.5]x_2 + [5.5,7.5]x_3 + [0.5,2.5]x_4 \\ Z_2 &= [2.5,4.5]x_1 + [0.5,2.5]x_2 + [1.5,3.5]x_3 + [9.5,11.5]x_4 \\ Z_3 &= [8.5,10.5]x_1 + [2.5,4.5]x_2 + [0.5,2.5]x_3 + [1.5,3.5]x_4 \\ \min W_1 &= [0.75,1.75]x_1 + [1.5,3.5]x_2 + [0.05,0.35]x_3 + [2.5,4.5]x_4 \\ W_2 &= [0.2,0.68]x_1 + [0.5,2.5]x_2 + [0.4,0.8]x_3 + [1.5,3.5]x_4 \end{aligned} \tag{6}$$

Subject to

$$[2.5,4.5]x_1 + [0.75,1.75]x_2 + [0.75,1.75]x_3 + [6.75,7.75]x_4 = [75,175]$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

According to MOLP Problem, problem (6) can be written as follows:

$$\begin{aligned} \max (Z_1)^L &= 1.5x_1 + 3.5x_2 + 5.5x_3 + 0.5x_4 \\ (Z_1)^C &= 5x_1 + 4.5x_2 + 6.5x_3 + 1.5x_4 \\ (Z_2)^L &= 2.5x_1 + 0.5x_2 + 1.5x_3 + 9.5x_4 \\ (Z_2)^C &= 3.5x_1 + 1.5x_2 + 2.5x_3 + 10.5x_4 \\ (Z_3)^L &= 8.5x_1 + 2.5x_2 + 0.5x_3 + 1.5x_4 \\ (Z_3)^C &= 9.5x_1 + 3.5x_2 + 1.5x_3 + 2.5x_4 \\ \min (W_1)^R &= 1.75x_1 + 3.5x_2 + 0.35x_3 + 4.5x_4 \\ (W_1)^C &= 1.25x_1 + 2.5x_2 + 0.2x_3 + 3.5x_4 \\ (W_2)^R &= 0.68x_1 + 2.5x_2 + 0.8x_3 + 3.5x_4 \\ (W_2)^C &= 0.44x_1 + 1.5x_2 + 0.6x_3 + 2.5x_4 \end{aligned} \tag{7}$$

Subject to

$$2.5x_1+0.75x_2+0.75x_3+6.75x_4=75, 4.5x_1+1.75x_2+1.75x_3+7.75x_4=175$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

Problem (7) using the min operator becomes

$$\max \delta$$

Subject to

$$\begin{aligned}
&1.5x_1+3.5x_2+5.5x_3+0.5x_4-200\delta \geq 350,\\
&5x_1+4.5x_2+6.5x_3+1.5x_4-200\delta \geq 450,\\
&2.5x_1+0.5x_2+1.5x_3+9.5x_4-100\delta \geq 50,\\
&3.5x_1+1.5x_2+2.5x_3+10.5x_4-100\delta \geq 150,\\
&8.5x_1+2.5x_2+0.5x_3+1.5x_4-200\delta \geq 50,\\
&9.5x_1+3.5x_2+1.5x_3+2.5x_4-200\delta \geq 150,\\
&1.75x_1+3.5x_2+0.35x_3+4.5x_4+315\delta \leq 350,\\
&1.25x_1+2.5x_2+0.2x_3+3.5x_4+230\delta \leq 250,\\
&0.68x_1+2.5x_2+0.8x_3+3.5x_4+170\delta \leq 250,\\
&0.44x_1+1.5x_2+0.6x_3+2.5x_4+90\delta \leq 150,\\
&0<\delta \leq 1; 2.5x_1+0.75x_2+0.75x_3+6.75x_4=75,\\
&4.5x_1+1.75x_2+1.75x_3+7.75x_4=175; x_1, x_2, x_3, x_4 \geq 0.
\end{aligned} \tag{8}$$

The solution of (8) is

$$\delta^*=0.5 \quad, x=(0,50,50,0),\ (Z_1)^L=962.5,\ (Z_1)^C=550\ ,(Z_2)^L=100\ ,(Z_2)^C=200,$$
$$(Z_3)^L=150,\ (Z_3)^C=250,\ (W_1)^R=192.5,\ (W_1)^C=135,\ (W_2)^R=165, (W_2)^C=105.$$

Problem (7) according to the average operator model (2) is

$$\max \bar{\delta}=1/10\left(\delta_1^{'}+\delta_1^{''}+\delta_2^{'}+\delta_2^{''}+\delta_3^{'}+\delta_3^{''}+\delta_1^{1}+\delta_1^{2}+\delta_2^{1}+\delta_2^{2}\right)$$

Subject to

$$\begin{aligned}
&1.5x_1+3.5x_2+5.5x_3+0.5x_4-200\delta_1^{'} \geq 350,\\
&5x_1+4.5x_2+6.5x_3+1.5x_4-200\delta_1^{''} \geq 450,\\
&2.5x_1+0.5x_2+1.5x_3+9.5x_4-100\delta_2^{'} \geq 50,\\
&3.5x_1+1.5x_2+2.5x_3+10.5x_4-100\delta_2^{''} \geq 150,\\
&8.5x_1+2.5x_2+0.5x_3+1.5x_4-200\delta_3^{'} \geq 50,\\
&9.5x_1+3.5x_2+1.5x_3+2.5x_4-200\delta_3^{''} \geq 150,
\end{aligned} \tag{9}$$

$$1.75x_1 + 3.5x_2 + 0.35x_3 + 4.5x_4 + 315\delta_1^1 \le 350,$$
$$1.25x_1 + 2.5x_2 + 0.2x_3 + 3.5x_4 + 230\delta_1^2 \le 250,$$
$$0.68x_1 + 2.5x_2 + 0.8x_3 + 3.5x_4 + 170\delta_2^1 \le 250,$$
$$0.44x_1 + 1.5x_2 + 0.6x_3 + 2.5x_4 + 90\delta_2^2 \le 150,$$
$$2.5x_1 + 0.75x_2 + 0.75x_3 + 6.75x_4 = 75, 4.5x_1 + 1.75x_2 + 1.75x_3 + 7.75x_4 = 175,$$
$$0 \le \delta_i^{'}, \delta_i^{''} \le 1,\ i = 1,2,3;\ 0 \le \delta_1^1, \delta_1^2, \delta_2^1, \delta_2^2 \le 1;\ x_1, x_2, x_3, x_4 \ge 0.$$

The solution of problem (9) is

$$\bar{\delta} = 0.7 \quad, x = (0,0,100,0),\ (Z_1)^L = 550,\ (Z_1)^C = 650\ ,(Z_2)^L = 150\ ,(Z_2)^C = 250,$$
$$(Z_3)^L = 50,\ (Z_3)^C = 150,\ (W_1)^R = 35,\ (W_1)^C = 20,\ (W_2)^R = 80,\ (W_2)^C = 60,$$
$$\delta_1^{'} = 1\ ,\ \delta_1^{''} = 1\ ,\delta_2^{'} = 1\ ,\ \delta_2^{''} = 1\ ,\delta_3^{'} = 0\ ,\ \delta_3^{''} = 0\ ,\delta_1^1 = 1\ ,\delta_2^1 = 1\ ,\delta_2^1 = 1\ ,\delta_2^2 = 0 \quad .$$

The two-phase approach model (3) corresponding to the problem (7) becomes

$$\max \bar{\delta} = 1/10\left(\delta_1^{'} + \delta_1^{''} + \delta_2^{'} + \delta_2^{''} + \delta_3^{'} + \delta_3^{''} + \delta_1^1 + \delta_1^2 + \delta_2^1 + \delta_2^2\right)$$

Subject to

$$\begin{aligned}
&1.5x_1 + 3.5x_2 + 5.5x_3 + 0.5x_4 - 200\delta_1^{'} \ge 350,\\
&5x_1 + 4.5x_2 + 6.5x_3 + 1.5x_4 - 200\delta_1^{''} \ge 450,\\
&2.5x_1 + 0.5x_2 + 1.5x_3 + 9.5x_4 - 100\delta_2^{'} \ge 50,\\
&3.5x_1 + 1.5x_2 + 2.5x_3 + 10.5x_4 - 100\delta_2^{''} \ge 150,\\
&8.5x_1 + 2.5x_2 + 0.5x_3 + 1.5x_4 - 200\delta_3^{'} \ge 50,\\
&9.5x_1 + 3.5x_2 + 1.5x_3 + 2.5x_4 - 200\delta_3^{''} \ge 150,\\
&1.75x_1 + 3.5x_2 + 0.35x_3 + 4.5x_4 + 315\delta_1^1 \le 350,\\
&1.25x_1 + 2.5x_2 + 0.2x_3 + 3.5x_4 + 230\delta_1^2 \le 250,\\
&0.68x_1 + 2.5x_2 + 0.8x_3 + 3.5x_4 + 170\delta_2^1 \le 250,\\
&0.44x_1 + 1.5x_2 + 0.6x_3 + 2.5x_4 + 90\delta_2^2 \le 150,\\
&0.5 \le \delta_i^{'}, \delta_i^{''} \le 1,\ i = 1,2,3;\ 0.5 \le \delta_1^1, \delta_1^2, \delta_2^1, \delta_2^2 \le 1,\\
&2.5x_1 + 0.75x_2 + 0.75x_3 + 6.75x_4 = 75, 4.5x_1 + 1.75x_2 + 1.75x_3 + 7.75x_4 = 175,\\
&x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{10}$$

The solution is

$$\bar{\delta} = 0.8 \quad, x = (0,0,100,0),\ (Z_1)^L = 550,\ (Z_1)^C = 650\ ,(Z_2)^L = 150\ ,(Z_2)^C = 250,$$
$$(Z_3)^L = 50,\ (Z_3)^C = 150,\ (W_1)^R = 35,\ (W_1)^C = 20,\ (W_2)^R = 80,\ (W_2)^C = 60,$$
$$\delta_1^{'} = 1\ ,\ \delta_1^{''} = 1\ ,\delta_2^{'} = 1\ ,\ \delta_2^{''} = 1\ ,\delta_3^{'} = 0\ ,\ \delta_3^{''} = 0\ ,\delta_1^1 = 1\ ,\delta_2^1 = 1\ ,\delta_2^1 = 1\ ,\delta_2^2 = 1.$$

6. Concluding Remarks

In the fuzzy multi-objective linear programming problem, firstly, the problem was converted into a deterministic problem based on the $\alpha-$levels of the fuzzy numbers. Then, a two-phase approach with equal weighted coefficients to the deterministic problem generated an $\alpha-$efficient solution. The major advantage of the proposed method is that proposed approach is as long as the weighted coefficients not necessary equal and generate an efficient solution. These results provide contributions for a social planner who tries to allocate resources efficiently.

Acknowledgments

The author would like to thank the anonymous referees for their valuable suggestions and helpful comments which improved the quality of this paper.

References

[1] Bitran, G. R. (1980). Linear multiple objective problems with interval coefficients. *Management science*, *26*(7), 694-706.
[2] Dubois, D. J. (1980). *Fuzzy sets and systems: Theory and applications*. Academic press.
[3] Garg, H. (2015). Multi-objective optimization problem of system reliability under intuitionistic fuzzy set environment using Cuckoo Search algorithm. *Journal of intelligent & fuzzy systems*, *29*(4), 1653-1669.
[4] Garg, H. (2018). Some arithmetic operations on the generalized sigmoidal fuzzy numbers and its application. *Granular computing*, *3*(1), 9-25.
[5] Garg, H. (2018). Arithmetic operations on generalized parabolic fuzzy numbers and its application. *Proceedings of the national academy of sciences, India section A: Physical sciences*, *88*(1), 15-26.
[6] Garg, H. (2018). Analysis of an industrial system under uncertain environment by using different types of fuzzy numbers. *International journal of system assurance engineering and management*, *9*(2), 525-538.
[7] Gasimov, R. R., & Yenilmez, K. (2002). Solving fuzzy linear programming problems with linear membership function. *Turkish journal of mathematics*, *26*(4), 375-396.
[8] Hamadameen, A. O. (2017). A novel technique for solving multiobjective fuzzy linear programming problems. *ARO-the scientific journal of Koya university*, *5*(1), 1-8.
[9] Kaufmann, A., & Gupta, M. M. (1988). *Fuzzy mathematical models in engineering and management science*. New York, NY, USA: Elsevier Science Inc.
[10] Singh, S. K., & Yadav, S. P. (2017). Intuitionistic fuzzy multi-objective linear programming problem with various membership functions. *Annals of operations research*, 1-15.
[11] Moore, R. E. (1979). *Methods and applications of interval analysis*. Siam.
[12] Philip, J. (1972). Algorithms for the vector maximization problem. *Mathematical programming*, *2*(1): 207-229.
[13] Rani, D., Gulati, T. R., & Garg, H. (2016). Multi-objective non-linear programming problem in intuitionistic fuzzy environment: optimistic and pessimistic view point. *Expert systems with applications*, *64*, 228-238.
[14] Sakawa, M., & Yano, H. (1985). Interactive decision making for multi-objective linear fractional programmingproblems with fuzzy parameters. *Cybernetics and system*, *16*(4), 377-394.
[15] Steuer, R. E. (1986). Multiple criteria optimization. *Theory, computation and applications*.
[16] Shaocheng, T. (1994). Interval number and fuzzy number linear programmings. *Fuzzy sets and systems*, *66*(3), 301-306.
[17] Veeramani, C., Duraisamy, C., & Nagoorgani, A. (2011). Solving fuzzy multi-objective linear programming problems with linear membership functions. *Australian journal of basic and applied sciences*, *5*(8), 1163-1171.
[18] Wang, F. H., & Wang, L. M. (1996). A decision making procedure of a fuzzy multiobjecttive linear problem. *Journal of the Chinese institute of industrial engineers, 13*(1), 1-10.
[19] Zadeh, L. A. (1965). Fuzzy sets. *Information control,* (8) 338- 353.
[20] Zimmermann, H. J. (1978). Fuzzy programming and linear programming with several objective functions. *Fuzzy sets and systems*, *1*(1), 45-55.