J. Appl. Res. Ind. Eng. Vol. 6, No. 2 (2019) 139–149

Journal of Applied Research on Industrial Engineering
www.journal-aprie.com

A New Approach to Solve Fully Fuzzy Linear Programming Problem

Seyed Hadi Nasseri**, Faranak Mahmoudi

Department of mathematics, University of Mazandaran, Babolsar, Iran.

PAPER INFO

Chronicle:
Received: 27 March 2019
Revised: 24 May 2019
Accepted: 27 June 2019

Keywords:
Fully Fuzzy Linear Programming.
Triangular Fuzzy Numbers.
Ranking Function.
Fuzzy Number.

ABSTRACT

Today, human decisions are more than ever based on information. But most of this information is not definitive, and in this situation, logical decision making is very difficult based on this uncertainty. Different methods are used to represent this uncertainty, including the fuzzy numbers. The fuzzy linear programming problem is one of the interesting concepts to be addressed in fuzzy optimization. Fully Fuzzy Linear Programming Problems (FFLP) are issues in which all parameters of the coefficients of the variables in the target functions, the coefficients of the variables in the constraints, the right-hand side of the constraints, and the decision variables in them are fuzzy. In this paper, we show that Definition 2.6 which is used by Ezzati et al. [1], failed to compare any arbitrary triangular fuzzy numbers. We demonstrate that their presented method is not well in general, thus the proposed method finds the fuzzy optimal solution of fully fuzzy linear programming problems by Ezzati et al. [1]. Then a new approach is proposed for solving this FFLP problem. An example is also presented to demonstrate the new method.

1. Introduction

Linear Programming (LP) is one of the most versatile, powerful, and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique helps us is linear programming. As a decision making tool, it has demonstrated its value in various field such as production, finance, marketing, research and development, and personal management. In conventional LP problems, it is assumed that the data have precise values. This means that the elements are crisp numbers, inequality is defined in the crisp sense, and objective function is a strict imperative. However, the observed values of the data in real-life problems are often imprecise because of incomplete or non-obtainable information. In such situations, fuzzy sets theory is an idea approach to handle imprecise in LP by generalizing the notion of membership in a set and this leads to the concept of fuzzy LP problems. Fuzzy Linear Programming (FLP) problems allow working with imprecise data and constraints, leading to more realistic models. They have often been used for solving a wide variety of problems in sciences and engineering. Fuzzy mathematical programming has been researched by a number of authors. One of the earliest works on fuzzy mathematics programming

* Corresponding author
E-mail address: nhadi57@gmail.com
DOI: 10.22105/jarie.2019.183391.1090

problems was presented by Tanaka et al. [2] based on the fuzzy decision framework of Bellman and Zadeh [3]. Since Tanaka et al. [2] have been a number of fuzzy LP models, in the literature, the fuzzy LP has been classified into different categories, depending on how imprecise parameters are modeled by subjective preference-based membership functions or possibility distributions. Numerous researchers have studied various properties of FLP problems and proposed different approaches for solving them. Because of existing different assumptions and sources of fuzziness in the parameters, the definition of FLP problem is not unique. Bector and Chandra [4] have classified FLP problems into four following Categories:

- **Type I.** LP problems with fuzzy inequalities and crisp objective function.
- **Type II.** LP problems with crisp inequalities and fuzzy objective function.
- **Type III.** LP problems with fuzzy inequalities and fuzzy objective function.
- **Type IV.** LP problems with fuzzy parameters.

The FLP problems involving fuzzy numbers for the decision variables, the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints, is called Fully Fuzzy Linear Programming (FFLP) problems. Lotfi et al. [5] considered the FFLP problems where all the parameters and variables were triangular fuzzy numbers. Kumar et al. [6] using arithmetic operations and definition of fuzzy equality, first converted each fuzzy equality constraint into several crisp constraints and then optimized the rank of fuzzy objective function over the obtained crisp feasible space. In contrast to most existing approaches which provide crisp solution, their method gives fuzzy optimal solution but also not preserve the form of non-negative fuzzy optimal solution and optimal objective function, then Najafi and Edalatpanah [7] noticed that Kumar et al. [6] method was missing one condition to guarantee the non-negativity of the fuzzy solution. Fully Fuzzy Linear Programming problem in which all the parameters and variables are considered as fuzzy numbers is an attractive topic for researchers Lotfi et al. [5], Kumar et al. [6], and Ezzati et al. [1]. Kumar et al. [6] proposed a new method to solve FFLP problem where all the constraints are equality.

In method [5], the parameters of fully fuzzy linear programming problem have been approximated to the nearest symmetric triangular fuzzy numbers. Then a fuzzy optimal approximation solution has been achieved by solving a Multi-Objective Linear Programming (MOLP) problem. In method [6], the linear ranking function has been used to convert the fuzzy objective function to crisp objective function. Bhardwaj and Kumar [8] shown that the fully fuzzy programming problems with inequality constraints cannot be transformed into fully fuzzy linear programming problems with equality constraints. And hence, the algorithm, proposed by Ezzati et al. [1] for solving fully fuzzy linear programming problems with equality constraints, cannot be used for finding the fuzzy optimal solution of fully fuzzy linear programming problems with inequality constraints.

Ezzati et al. [1] introduced a definition to comparing triangular fuzzy numbers and the use of it. They proposed a new algorithm to find the optimal solution of fully fuzzy linear programming problem. Based on a new lexicographic ordering on triangular fuzzy numbers, a novel algorithm is proposed to solve the FFLP problem by converting it to its equivalent, a MOLP problem, and then it is solved by the lexicographic method. In this paper, we study Ezzati et al. [1] by using some definitions and numerical examples that are shown that Ezzati's definition for comparing triangular fuzzy numbers is not hold for each fuzzy numbers. Examples are provided to prove this claim and using of this fact proposes a method to converting fully fuzzy problem into crisp linear programming problem, which is improved.

The rest of this paper is organized as follows: In Section 2, we review the basic definitions and results of fuzzy sets and some related topics. Section 3 gives the definition for comparing fuzzy numbers and then, we propose numerical example. Furthermore, we introduce a new method for solving FFLP problem in Section 3. In Section 4, an application of the method is described in FLP problems and finally the conclusions are discussed in Section 5.

2. Preliminaries

In this section, we begin with some basic definitions, arithmetic operations of fuzzy numbers, and an existing ranking approach for comparing fuzzy numbers will be used in the rest of this paper.

Definition 1. Let $\mathbb{R}$ denotes a universal set. Then, a fuzzy subset $\tilde{A}$ of $\mathbb{R}$ is defined by its membership function $\mu_{\tilde{A}}: \mathbb{R} \to [0,1]$, which assigns a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$,

To each element $x \in \mathbb{R}$, where the value of $\mu_{\tilde{A}}(x)$ at x shows that grade of membership of x in $\tilde{A}$.

A fuzzy subset $\tilde{A}$ can be characterized as a set of ordered pairs of element x and grade $\mu_{\tilde{A}}(\mathbb{R})$ and is often written $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in \mathbb{R}\}$; the class of fuzzy sets on R is denoted by $TF(\mathbb{R})$.

Definition 2. The α-cut or α-level set of a fuzzy set is a crisp set defined by $A_\alpha = \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) > \alpha\}$.

Definition 3. A fuzzy set A on $\mathbb{R}$ is convex, if for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, we have $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$.

Definition 4. A fuzzy number $\tilde{A} = (x_1, y_1, z_1) = (x_1^l, y_1^c, z_1^u)$ is said to be a triangular fuzzy number if its membership function is given as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - x_1}{y_1 - x_1}, & x_1 \leq x \leq y_1, \\ \dfrac{x - z_1}{y_1 - z_1}, & y_1 \leq x \leq z_1, \\ 0, & O.W. \end{cases} \tag{1}$$

Definition 5. A triangular fuzzy number $\tilde{A} = (x_1, y_1, z_1)$ is said to be a non- negative triangular fuzzy number, if and only if $x_1 \geq 0$. The set of all these triangular fuzzy numbers is denoted by $TF(\mathbb{R})^+$.

Definition 6. An effective approach for ordering the elements of $\mathcal{F}(\mathbb{R})$ is to define a ranking function $\mathcal{R}: \mathcal{F}(\mathbb{R}) \to R$ which maps each fuzzy number into the real line, where a natural order exists [9].

We define orders on $\mathcal{F}(\mathbb{R})$ by

$\tilde{A} \geq_{\mathcal{R}} \tilde{B}$ *if and only if* $\mathcal{R}(\tilde{A}) \geq \mathcal{R}(\tilde{B})$,

$\tilde{A} >_{\mathcal{R}} \tilde{B}$ *if and only if* $\mathcal{R}(\tilde{A}) > \mathcal{R}(\tilde{B})$,

$\tilde{A} =_{\mathcal{R}} \tilde{B}$ *if and only if* $\mathcal{R}(\tilde{A}) = \mathcal{R}(\tilde{B})$,

where $\tilde{A}$ *and* $\tilde{B}$ are in $\mathcal{F}(\mathbb{R})$.

A ranking function $\mathcal{R}$ is said to be a linear ranking function if $\mathcal{R}(k\tilde{A} + \tilde{B}) = k\mathcal{R}(\tilde{A}) + \mathcal{R}(\tilde{B})$ for any $k \in \mathbb{R}$.

Remark 1. As well as given in [9], we use ranking function for triangular fuzzy number $\tilde{A} = (x_1, y_1, z_1)$ as follows:

$$\mathcal{R}(\tilde{A}) = \frac{1}{4}(x_1 + 2y_1 + z_1)\,. \tag{2}$$

Definition 7. Two triangular fuzzy numbers $\tilde{A} = (x_1, y_1, z_1)$ and $\tilde{B} = (x_2, y_2, z_2)$ are said to be equal, $\tilde{A} = \tilde{B}$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

Definition 8. Let $\tilde{A} = (x_1, y_1, z_1)$ and $\tilde{B} = (x_2, y_2, z_2)$ be two triangular fuzzy numbers. Then, arithmetic operation on these fuzzy numbers can be defined as follows:

- Addition: $\tilde{A} \oplus \tilde{B} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$,
- Subtraction: $\tilde{A} \ominus \tilde{B} = (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$,
- $-\tilde{A} = -(x_1, y_1, z_1) = (-z_1, -y_1, -x_1)$,

- Multiplication: if $\tilde{B}$ be a non- negative triangular fuzzy number then:

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (x_1x_2, y_1y_2, z_1z_2), & x_1 \geq 0, \\ (x_1z_2, y_1y_2, z_1z_2), & x_1 < 0\,, z_1 \geq 0, \\ (x_1z_2, y_1y_2, z_1x_2), & z_1 < 0. \end{cases}$$

Definition 9. The FLP problem is said to be an FFLP problem if all the parameters and variables are considered as fuzzy numbers. In recent years, some researchers such as [5, 8] were interested in the FFLP problems, and some solution methods have been obtained to the fully fuzzy systems and the FFLP problems [10, 11, 12]. These problems can be divided in two categories: (1) problems with inequality constraints; (2) problems with equality constraints. Let an FFLP problem be as follows:

$$max(min) \qquad \sum_{j=1}^{n} \tilde{C}_j^T \otimes \tilde{x}_j \tag{3}$$

$$s.t. \qquad \begin{cases} \sum_{j=1}^{n} \tilde{A}_{ij} \otimes \tilde{x}_j \leq (=, \geq) \tilde{b}_i, & i = 1,2,\dots,m, \\ \tilde{x}_j & is\ non-negative\ fuzy\ umber \end{cases}$$

where $\tilde{C}_j^T = [\tilde{c}_j^T]_{1\times n}$, $\tilde{b}_i = [\tilde{b}_i]_{m\times 1}$, $\tilde{A}_{ij} = [\tilde{A}_{ij}]_{m\times n}$and $\tilde{x}_j = [\tilde{x}_j]_{1\times n}$. Let us assume that all fuzzy numbers are triangular. Thus $\tilde{C}_j^T$, $\tilde{b}_i, \tilde{A}_{ij}$ and $\tilde{x}_j$, are represented as $\tilde{C}_j^T = (C_j^l, C_j^c, C_j^u)^T$, $\tilde{b}_i = (b_i^l, b_i^c, b_i^u)$, $\tilde{A}_{ij} = (A_{ij}^l, A_{ij}^c, A_{ij}^u)$ and $\tilde{x}_j = (x_j^l, x_j^c, x_j^u)$, respectively. Now by substituting these triangular fuzzy numbers in problem (3), the FFLP problem (3) may be written as:

$$max(min) \qquad \sum_{j=1}^{n} (C_j^l, C_j^c, C_j^u)^T \otimes (x_j^l, x_j^c, x_j^u) \tag{4}$$

$$s.t. \qquad \begin{cases} \sum_{j=1}^{n} (A_{ij}^l, A_{ij}^c, A_{ij}^u) \otimes (x_j^l, x_j^c, x_j^u) \leq (=, \geq)(b_i^l, b_i^c, b_i^u), & i = 1,2,\dots,m, \\ (x_j^l, x_j^c, x_j^u) & is\ non-negative\ fuzy\ umber \end{cases}$$

3. Proposed Method to Solve FFLP Problem

Ezzati et al. [1] presented solution algorithm for all fully fuzzy problems with non-negative triangular fuzzy variables, which this assumption limits the algorithm to solve specific problems and reduce the generality of the method to solve all fully fuzzy problems. Bhardwaj and Kumar [8] pointed out an example that this method is not efficient to solve a fully fuzzy programming problems with inequality constraints. According to Definition 6, the optimal solution of problem, obtained from Ezzati's method, is not a feasible fuzzy solution of the original problem.

Ezzati et al. [1] in Definition 6 of, proposed a method for comparing any arbitrary fuzzy numbers that by use of yager's ranking function in definition 6 and Remark 1, and solving the following counterexamples showed that their definition for all fuzzy numbers will not be right.

For the continuing of the discussion, we mention the ranking definition which is defined by Ezzati et al. [1] and then we can explicitly say this method of evaluation may produce inaccurate results, as shown in Fig. 1 and the current discussion.

Ezzati et al. [1] in Definition 6 expressed that if $\tilde{A} = (x_1, y_1, z_1)$ and $\tilde{B} = (x_2, y_2, z_2)$ be two triangular fuzzy numbers then say that $\tilde{A} \prec \tilde{B}$, if and only if:

- $y_1 < y_2$ or
- $y_1 = y_2$ and $(z_1 - x_1) > (z_2 - x_2)$ or
- $y_1 = y_2$ and $(z_1 - x_1) = (z_2 - x_2)$ and $(x_1 + z_1) < (x_2 + z_2)$.

Counterexample. Consider two triangular fuzzy numbers $\tilde{A} = (0,1,1.5)$ and $\tilde{B} = (0,1,1.6)$ then according to Ezzati's definition in this example we have that 1=1, 1.6> 1.5 then $\tilde{B} < \tilde{A}$ while due to the Definition 5 of this paper (Yagre's ranking function), $\mathcal{R}(\tilde{A}) = \frac{1}{4}(0 + 2 + 1.5) = 1.125$ and $\mathcal{R}(\tilde{B}) = \frac{1}{4}(0 + 2 + 1.6) = 1.115$ since $\mathcal{R}(\tilde{A}) < \mathcal{R}(\tilde{B})$ then $\tilde{A} < \tilde{B}$.

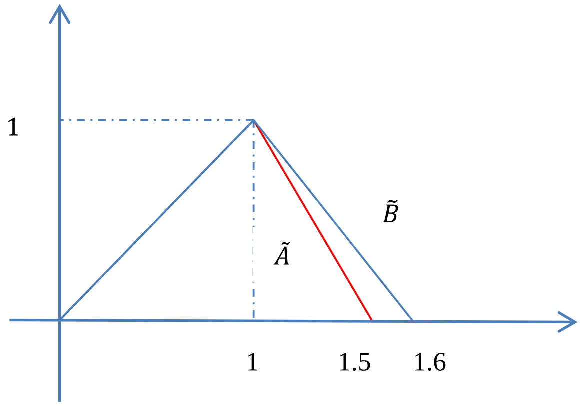

Fig. 1. *Showing two triangular fuzzy numbers* $\tilde{A} = (0,1,1.5)$ *and* $\tilde{B} = (0,1,1.6)$.

Also, according to the available methods for comparing two triangular fuzzy numbers, the comparative table shows the ranking methods of two triangular fuzzy numbers and then results that $\tilde{A} < \tilde{B}$.

Select Yager's ranking method is arbitrary from this table.

Table 1. *Ranking methods of two triangular fuzzy numbers.*

	Adamo method	Li and Lee method	Robens method	Cheng method	Yager method	Ezzati method
$\tilde{A}=(0,1,1.5)$	$AD(\tilde{A}) = 1.5-0.5\alpha$	$\overline{x}_u(\tilde{A})=\frac{2.5}{3}$	$R(\tilde{A})=\frac{2.5}{2}$	$CH(\tilde{A})=\frac{2.5}{6}$	$Y(\tilde{A})=\frac{2.5}{4}$	–
$\tilde{B}=(0,1,1.6)$	$AD(\tilde{B}) = 1.6-0.6\alpha$	$\overline{x}_u(\tilde{B})=\frac{2.6}{3}$	$R(\tilde{B})=\frac{2.6}{2}$	$CH(\tilde{B})=\frac{2.6}{6}$	$Y(\tilde{B})=\frac{2.6}{4}$	–
result	$\tilde{A}<\tilde{B}$	$\tilde{A}<\tilde{B}$	$\tilde{A}<\tilde{B}$	$\tilde{A}<\tilde{B}$	$\tilde{A}<\tilde{B}$	$\tilde{A}>\tilde{B}$

Ezzati et al. [1] represented an algorithm to solve each fully fuzzy problem based on a comparing method. We have shown that their comparing method and definition is not be true for all fuzzy numbers and by disproving Ezzati's method. In this way, we convert the objective function into crisp objectives as follows.

Consider the following FFLP problem (5),

$$\max \quad \sum_{j=1}^{n}\left(C_j^l,C_j^c,C_j^u\right)^T\otimes\left(x_j^l,x_j^c,x_j^u\right) \tag{5}$$
$$s.t.\quad \begin{cases}\sum_{j=1}^{n}\left(A_{ij}^l,A_{ij}^c,A_{ij}^u\right)\otimes\left(x_j^l,x_j^c,x_j^u\right)\le\left(b_i^l,b_i^c,b_i^u\right), & i=1,2,\ldots,m,\\ \left(x_j^l,x_j^c,x_j^u\right) & is\ non-negative\ fuzy\ umber\end{cases}$$

Assuming $\left(A_{ij}^l,A_{ij}^c,A_{ij}^u\right)\otimes\left(x_j^l,x_j^c,x_j^u\right)=\left(m_{ij}^l,m_{ij}^c,m_{ij}^u\right)$ the FFLP problem may be written as follows:

$$\max \quad \sum_{j=1}^{n}\left(C_j^l,C_j^c,C_j^u\right)^T\otimes\left(x_j^l,x_j^c,x_j^u\right) \tag{6}$$
$$s.t.\quad \begin{cases}\sum_{j=1}^{n}\left(m_{ij}^l,m_{ij}^c,m_{ij}^u\right)\le\left(b_i^l,b_i^c,b_i^u\right), & i=1,2,\ldots,m,\\ \left(x_j^l,x_j^c,x_j^u\right) & is\ non-negative\ fuzy\ umber\end{cases}$$

Using arithmetic operations defined in Definitions 6 and 8, the fully fuzzy linear programming problem (6), is converted into the following problem:

$$\max \quad \mathcal{R}\left(\sum_{j=1}^{n}\left(C_j^l,C_j^c,C_j^u\right)^T\otimes\left(x_j^l,x_j^c,x_j^u\right)\right) \tag{7}$$
$$s.t.\quad \begin{cases}\sum_{j=1}^{n} m_{ij}^l\le b_i^l, & i=1,2,\ldots,m,\\ \sum_{j=1}^{n} m_{ij}^c\le b_i^c, & i=1,2,\ldots,m,\\ \sum_{j=1}^{n} m_{ij}^u\le b_i^u, & i=1,2,\ldots,m,\\ x_j^c-x_j^l\ge 0, x_j^u-x_j^c\ge 0, x_j^l\ge 0. & \end{cases}$$

Using ranking function given in Remark 1 for the objective function, the FLP problem (7) can be rewritten as follows:

$$max \quad \frac{1}{4}\left(\sum_{j=1}^{n} C_j^l x_j^l + 2\sum_{j=1}^{n} C_j^c x_j^c + \sum_{j=1}^{n} C_j^u x_j^u\right) \tag{8}$$

$$s.t. \quad \begin{cases} \sum_{j=1}^{n} m_{ij}^l \leq b_i^l, & i = 1,2,\dots,m, \\ \sum_{j=1}^{n} m_{ij}^c \leq b_i^c, & i = 1,2,\dots,m, \\ \sum_{j=1}^{n} m_{ij}^u \leq b_i^u, & i = 1,2,\dots,m, \\ x_j^c - x_j^l \geq 0, x_j^u - x_j^c \geq 0, x_j^l \geq 0. \end{cases}$$

Now, we find the optimal solution x_j^l, x_j^c and x_j^u by solving the problem (8), and by putting the value of x_j^l, x_j^c and x_j^u in $\tilde{x}_j = (x_j^l, x_j^c, x_j^u)$ obtain the fuzzy optimal solution, then by substitute $\tilde{x}_j$ in $\sum_{j=1}^{n} \tilde{C}_j^T \otimes \tilde{x}_j$ find the optimal value of problem (5).

3. Numerical Discussion

In this section, we are going to explore the solving process which is introduced in the last section for the extended model by an illustrative example.

Example 1. Consider the FFLP problem and solve it by the proposed method as follows:

$$max \quad (10,15,17)\otimes\tilde{x}_1\oplus(10,16,20)\otimes\tilde{x}_2\oplus(10,14,17)\otimes\tilde{x}_3\oplus(10,12,14)\otimes\tilde{x}_4 \tag{9}$$

$$s.t. \quad (8,10,13)\otimes\tilde{x}_1\oplus(10,11,13)\otimes\tilde{x}_2\oplus(9,12,13)\otimes\tilde{x}_3\oplus(11,15,17)\otimes\tilde{x}_4 = (271.75,411.75,573.75)$$

$$(11,14,16)\otimes\tilde{x}_1\oplus(14,18,19)\otimes\tilde{x}_2\oplus(14,17,20)\otimes\tilde{x}_3\oplus(13,14,18)\otimes\tilde{x}_4 = (385.5,539.5,759.5).$$

where $\tilde{x}_j$, $j = 1,2,3,4$, is non-negative fuzzy number.

Now, let us consider $\tilde{x}_j = (x_j^l, x_j^c, x_j^u)$. Then the FFLP problem (9), is rewritten as follows:

$$max \quad (10,15,17)\otimes(x_1^l, x_1^c, x_1^u)\oplus(10,16,20)\otimes(x_2^l, x_2^c, x_2^u)\oplus(10,14,17)\otimes(x_3^l, x_3^c, x_3^u) \tag{10}$$

$$\oplus(10,12,14)\otimes(x_4^l, x_4^c, x_4^u)$$

$$s.t. \quad (8,10,13)\otimes(x_1^l, x_1^c, x_1^u)\oplus(10,11,13)\otimes(x_2^l, x_2^c, x_2^u)\oplus(9,12,13)\otimes(x_3^l, x_3^c, x_3^u)$$

$$\oplus(11,15,17)\otimes(x_4^l, x_4^c, x_4^u) = (271.75,411.75,573.75)$$

$$(11,14,16)\otimes(x_1^l, x_1^c, x_1^u)\oplus(14,18,19)\otimes(x_2^l, x_2^c, x_2^u)\oplus(14,17,20)\otimes(x_3^l, x_3^c, x_3^u)$$

$$\oplus(13,14,18)\otimes(x_4^l,x_4^c,x_4^u)=(385.5,539.5,759.5).$$

$\tilde{x}_j$, $j=1,2,3,4$, is non-negative fuzzy number.

By use of ranking function that is mentioned in Definition 4, the FFLP problem (10), is written as follows:

$$max \quad \mathcal{R}\begin{pmatrix}(10,15,17)\otimes(x_1^l,x_1^c,x_1^u)\oplus(10,16,20)\otimes(x_2^l,x_2^c,x_2^u)\oplus(10,14,17)\otimes(x_3^l,x_3^c,x_3^u)\oplus\\(10,12,14)\otimes(x_4^l,x_4^c,x_4^u)\end{pmatrix} \tag{11}$$

$$s.t. \quad (8,10,13)\otimes(x_1^l,x_1^c,x_1^u)\oplus(10,11,13)\otimes(x_2^l,x_2^c,x_2^u)\oplus(9,12,13)\otimes(x_3^l,x_3^c,x_3^u)$$

$$\oplus(11,15,17)\otimes(x_4^l,x_4^c,x_4^u)=(271.75,411.75,573.75)$$

$$(11,14,16)\otimes(x_1^l,x_1^c,x_1^u)\oplus(14,18,19)\otimes(x_2^l,x_2^c,x_2^u)\oplus(14,17,20)\otimes(x_3^l,x_3^c,x_3^u)$$

$$\oplus(13,14,18)\otimes(x_4^l,x_4^c,x_4^u)=(385.5,539.5,759.5).$$

$\tilde{x}_j$, $j=1,2,3,4$, is non-negative fuzzy number.

Then, whit regard to problem (11) and the fuzzy arithmetic which is defined in Section 2, the problem (11) is rewrote as follows:

$$max\frac{1}{4}\Big((10x_1^l+10x_2^l+10x_3^l+10x_4^l)+(30x_1^c+32x_2^c+28x_3^c+24x_4^c) + (17x_1^u+20x_2^u+17x_3^u+14x_4^u)\Big) \tag{12}$$

$$s.t. \quad 8x_1^l+10x_2^l+9x_3^l+11x_4^l=271.75$$

$$10x_1^c+11x_2^c+12x_3^c+15x_4^c=411.75$$

$$13x_1^u+13x_2^u+13x_3^u+17x_4^u=573.75$$

$$11x_1^l+14x_2^l+14x_3^l+13x_4^l=385.5$$

$$14x_1^c+18x_2^c+17x_3^c+14x_4^c=539.5$$

$$16x_1^u+19x_2^u+20x_3^u+18x_4^u=759.5$$

$$x_1^l,x_1^c,x_1^u,x_2^l,x_2^c,x_2^u,x_3^l,x_3^c,x_3^u,x_4^l,x_4^c,x_4^u\geq 0.$$

The optimal solution of above mentioned linear programming problem is achieved as follows:

$$\tilde{X}^*=\begin{cases}\tilde{x}_1^*=((x_1^*)^l,(x_1^*)^C,(x_1^*)^u)=(25.5,31.94,26.35),\\ \tilde{x}_2^*=((x_2^*)^l,(x_2^*)^C,(x_2^*)^u)=(0,0,17.78),\\ \tilde{x}_3^*=((x_3^*)^l,(x_3^*)^C,(x_3^*)^u)=(7.5,0,0),\\ \tilde{x}_4^*=((x_4^*)^l,(x_4^*)^C,(x_4^*)^u)=(0,6.6,0).\end{cases}$$

Now, the optimal value of objective function can be obtained. Therefore, the optimal value of problem may be written as follows:

$$\tilde{C}^T\tilde{x}^* = ((C^T x^*)^l, (C^T x^*)^C, (C^T x^*)^u) = \left(\sum_{j=1}^4 \left(C_j x_j^*\right)^l, \sum_{j=1}^4 \left(C_j x_j^*\right)^C, \sum_{j=1}^4 \left(C_j x_j^*\right)^u\right) = (330,558.3,803.55),$$

Now, using Ezzati's method, the optimal solution and optimal value of objective function are given as follows:

$$\tilde{X}^* = \begin{cases} \tilde{x}_1^* = ((x_1^*)^l, (x_1^*)^C, (x_1^*)^u) = (17.27, 17.27, 17.27), \\ \tilde{x}_2^* = ((x_2^*)^l, (x_2^*)^C, (x_2^*)^u) = (2.16, 2.16, 2.16), \\ \tilde{x}_3^* = ((x_3^*)^l, (x_3^*)^C, (x_3^*)^u) = (4.64, 9.97, 16.36), \\ \tilde{x}_4^* = ((x_4^*)^l, (x_4^*)^C, (x_4^*)^u) = (6.36, 6.36, 6.36), \end{cases}$$

and putting the fuzzy optimal solution in the objective function of the problem gives

$$\left(\tilde{C}^T\tilde{x}^*\right)_{Ezzati's\ method} = \left(\sum_{j=1}^4 \left(C_j x_j^*\right)^l, \sum_{j=1}^4 \left(C_j x_j^*\right)^C, \sum_{j=1}^4 \left(C_j x_j^*\right)^u\right) = (304.58, 509.79, 704.37).$$

By comparing the results of proposed method in this paper with Ezzati's method [1], we can conclude that our result is more reliable, since:

$$(304.58, 509.79, 704.37) = \left(\tilde{C}^T\tilde{x}^*\right)_{Ezzati's\ method} < \left(\tilde{C}^T\tilde{x}^*\right)_{proposed\ \ method} = (330,558.3,803.55),$$

Example 2. Consider the following FFLP problem. Ezzati et al. [1] solved this problem, Bhardwaj, B., & Kumar [8] showed that this problem was not feasible with Ezati's proposed method.

We are now solving this problem by our proposed method and show that the problem will be solved and has feasible solution.

$$max\ \ ((5,7,8)\tilde{x}_{11} + (3,5,6)\tilde{x}_{12} + (4,8,9)\tilde{x}_{13} + (3,5,7)\tilde{x}_{21} + (4,7,8)\tilde{x}_{22} + (8,9,10)\tilde{x}_{23}$$

$$+(7,10,11)\tilde{x}_{31} + (6,8,10)\tilde{x}_{32} + (4,7,8)\tilde{x}_{33} + (4,6,8)\tilde{x}_{41} + (3,5,7)\tilde{x}_{42} + (7,9,11)\tilde{x}_{43}) \tag{13}$$

$$\text{s.t.}\ \ \sum_{i=1}^4 \sum_{j=1}^3 \tilde{x}_{ij} = (25,30,40),$$

$$\sum_{j=1}^3 \tilde{x}_{1j} \geq (2,3,5), \qquad \sum_{j=1}^3 \tilde{x}_{2j} \geq (4,5,6),$$

$$\sum_{j=1}^3 \tilde{x}_{3j} \geq (5,8,9)\ \ , \sum_{j=1}^3 \tilde{x}_{4j} \geq (7,8,14),$$

$$\tilde{x}_{11} \leq (4,6,7), \quad \tilde{x}_{12} \leq (3,5,6), \quad \tilde{x}_{13} \leq (8,9,10),$$

$$\tilde{x}_{21} \leq (5,7,8), \quad \tilde{x}_{22} \leq (8,10,11), \quad \tilde{x}_{23} \leq (3,4,5),$$

$$\tilde{x}_{31} \leq (4,5,7), \quad \tilde{x}_{32} \leq (2,3,6), \quad \tilde{x}_{33} \leq (4,7,9),$$

$$\tilde{x}_{41} \leq (4,6,7), \quad \tilde{x}_{42} \leq (4,5,9), \quad \tilde{x}_{43} \leq (2,5,4),$$

where $\tilde{x}_{ij}$, $i = 1,2,3,4,\ j = 1,2,3$, is non-negative fuzzy number.

By use of our proposed method the above problem is rewrote as follow:

$$max\frac{1}{4}\Big((5x_{11}^l+3x_{12}^l+4x_{13}^l+3x_{21}^l+4x_{22}^l+8x_{23}^l+7x_{31}^l+6x_{32}^l+4x_{33}^l+4x_{41}^l+3x_{42}^l+7x_{43}^l)$$
$$+2(7x_{11}^c+5x_{12}^c+8x_{13}^c+5x_{21}^c+7x_{22}^c+9x_{23}^c+10x_{31}^c+8x_{32}^c+7x_{33}^c+6x_{41}^c+5x_{42}^c+9x_{43}^c)$$
$$+(8x_{11}^u+6x_{12}^u+9x_{13}^u+7x_{21}^u+8x_{22}^u+10x_{23}^u+11x_{31}^u+10x_{32}^u+8x_{33}^u+8x_{41}^u+7x_{42}^u+11x_{43}^u)\Big)$$

$$
\begin{aligned}
s.t.\quad & x_{11}^l+x_{12}^l+x_{13}^l+x_{21}^l+x_{22}^l+x_{23}^l+x_{31}^l+x_{32}^l+x_{33}^l+x_{41}^l+x_{42}^l+x_{43}^l=25,\\
& x_{11}^c+x_{12}^c+x_{13}^c+x_{21}^c+x_{22}^c+x_{23}^c+x_{31}^c+x_{32}^c+x_{33}^c+x_{41}^c+x_{42}^c+x_{43}^c=30,\\
& x_{11}^u+x_{12}^u+x_{13}^u+x_{21}^u+x_{22}^u+x_{23}^u+x_{31}^u+x_{32}^u+x_{33}^u+x_{41}^u+x_{42}^u+x_{43}^u=40,\\
& x_{11}^l+x_{12}^l+x_{13}^l\geq 2,\\
& x_{11}^c+x_{12}^c+x_{13}^c\geq 3,\\
& x_{11}^u+x_{12}^u+x_{13}^u\geq 5,\\
& x_{21}^l+x_{22}^l+x_{23}^l\geq 4,\\
& x_{21}^c+x_{22}^c+x_{23}^c\geq 5,\\
& x_{21}^u+x_{22}^u+x_{23}^u\geq 8,\\
& x_{31}^l+x_{32}^l+x_{33}^l\geq 5,\\
& x_{31}^c+x_{32}^c+x_{33}^c\geq 8,\\
& x_{31}^u+x_{32}^u+x_{33}^u\geq 9,\\
& x_{41}^l+x_{42}^l+x_{43}^l\geq 7,\\
& x_{41}^c+x_{42}^c+x_{43}^c\geq 8,\\
& x_{41}^u+x_{42}^u+x_{43}^u\geq 14,\\
& x_{11}^l\leq 4,\quad x_{12}^l\leq 3,\quad x_{13}^l\leq 8,\\
& x_{11}^c\leq 6,\quad x_{12}^c\leq 5,\quad x_{13}^c\leq 9,\\
& x_{11}^u\leq 7,\quad x_{12}^u\leq 6,\quad x_{13}^u\leq 10,\\
& x_{21}^l\leq 5,\quad x_{22}^l\leq 8,\quad x_{23}^l\leq 3,\\
& x_{21}^c\leq 7,\quad x_{22}^c\leq 10,\quad x_{23}^c\leq 4,\\
& x_{21}^u\leq 8,\quad x_{22}^u\leq 11,\quad x_{23}^u\leq 5,\\
& x_{31}^l\leq 4,\quad x_{32}^l\leq 2,\quad x_{33}^l\leq 4,\\
& x_{31}^c\leq 5,\quad x_{32}^c\leq 3,\quad x_{33}^c\leq 7,\\
& x_{31}^u\leq 7,\quad x_{32}^u\leq 6,\quad x_{33}^u\leq 9,\\
& x_{41}^l\leq 4,\quad x_{42}^l\leq 4,\quad x_{43}^l\leq 2,\\
& x_{41}^c\leq 6,\quad x_{42}^c\leq 5,\quad x_{43}^c\leq 4,\\
& x_{41}^u\leq 7,\quad x_{42}^u\leq 9,\quad x_{43}^u\leq 5,
\end{aligned}
$$

where $\tilde{x}_{ij}$, $i=1,2,3,4,\ j=1,2,3$, is non-negative fuzzy number.

By solving above mentioned problem and by use of Lingo 17.0 software, we have:

$$
\tilde{X}^*=\begin{cases}
\tilde{x}_{11}^*=(4,0,0), & \tilde{x}_{12}^*=(0,0,0),\\
\tilde{x}_{13}^*=(0,3,5), & \tilde{x}_{21}^*=(0,0,0),\\
\tilde{x}_{22}^*=(5,1,1), & \tilde{x}_{23}^*=(3,4,5),\\
\tilde{x}_{31}^*=(4,5,7), & \tilde{x}_{32}^*=(2,3,3),\\
\tilde{x}_{33}^*=(0,0,0), & \tilde{x}_{41}^*=(4,0,0),\\
\tilde{x}_{42}^*=(1,0,0), & \tilde{x}_{43}^*=(2,14,19).
\end{cases}
$$

and putting the fuzzy optimal solution, in the objective function of the problem and comparing the results of proposed method in this paper with Ezzati's method [1], we can conclude that our result is more reliable, since:

$$(133, 245, 362) = \left(\tilde{C}^T\tilde{x}^*\right)_{Ezzati's\ method} < \left(\tilde{C}^T\tilde{x}^*\right)_{proposed\ \ method} = (137{,}267{,}419).$$

5. Conclusions

In this paper, we studied one of the new comparing definition and then also considered a solving method in [1] for solving fully fuzzy linear programming problems. We showed that this method is not correct generally. Furthermore, a new method was provided in this paper. Bhardwaj and Kumar [8] concluded that finding a feasible solution of the problem is not based on Definition 6 and thus becomes a matter of infeasible problem. We showed that this definition does not apply to all arbitrary fuzzy numbers and thus, there is a discussion as a result of Kumar's. By the proposed approach, the problem in solving the fully fuzzy linear model with inequality constraints was resolved .On the other hand, by proving that the definition expressed by Ezzati is not available for any arbitrary fuzzy numbers, then its extension to all numbers, and then to use it in all-fuzzy problems with equal and inequal constraints, and convert those to certain issues are not useful.

Acknowledgment

The authors would like to appreciate from the anonymous referees who offered their valuable comments to improve the earlier version of this study.

References

[1] Ezzati, R., Khorram, E., & Enayati, R. (2015). A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem. *Applied mathematical modelling*, *39*(12), 3183-3193.
[2] Tanaka, H., Okuda, T., & Asai, K. (1974). On fuzzy mathematical programming. *Jornal of cybern. 3*(4), 37–46.
[3] Bellman, R. E., & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. *Management science*, *17*(4), B-141.
[4] Bector, C. R., & Chandra, S. (2005). *Fuzzy mathematical programming and fuzzy matrix games* (Vol. 169). Berlin: Springer.
[5] Lotfi, F. H., Allahviranloo, T., Jondabeh, M. A., & Alizadeh, L. (2009). Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. *Applied mathematical modelling*, *33*(7), 3151-3156.
[6] Kumar, A., Kaur, J., & Singh, P. (2011). A new method for solving fully fuzzy linear programming problems. *Applied mathematical modelling*, *35*(2), 817-823.
[7] Najafi, H. S., & Edalatpanah, S. A. (2013). A note on "A new method for solving fully fuzzy linear programming problems". *Applied mathematical modelling*, *37*(14-15), 7865-7867.
[8] Bhardwaj, B., & Kumar, A. (2015). A note on "A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem". *Applied mathematical modelling*, *39*(19), 5982-5985.
[9] Nasseri, S. H., & Mahdavi-Amiri, N. (2009). Some duality results on linear programming problems with symmetric fuzzy numbers. *Fuzzy information and engineering*, *1*(1), 59-66.