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Fully Fuzzy Transportation Problems with Pentagonal and Hexagonal Fuzzy Numbers

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Abstract

The aim of this paper is to introduce a formulation of fully fuzzy transportation problems involving pentagonal and hexagonal fuzzy numbers for the transportation costs and values of supplies and demands. We introduce new technique for improve methods for solving the fully fuzzy transportation problems with parameters given as the pentagonal and hexagonal fuzzy numbers. Algorithms are proposed to find the non-negative fuzzy optimal solution of fully fuzzy transportation problems with parameters given as pentagonal and hexagonal fuzzy numbers. This technique is also best optimal solution in the literature and illustrated with numerical examples.

Keywords: Interval numbers, Pentagonal fuzzy numbers, Hexagonal fuzzy numbers, Fully fuzzy transportation problem.

1 | Introduction

Transportation problem is an important network structured linear programming problem that arises in several contexts and received a great deal of attention in the literature. Transportation problem can be used for a wide variety of situations such as production, investment, plant location, inventory control, employment scheduling and many others. In general, transportation problems are solved with the assumptions that the transportation costs, supplies at sources and demands at destinations are specified in a precise way i.e., in crisp environment. However, in many cases the decision maker may not able to get precise values for the decision parameters for the transportation problem. If the nature of the information is vague, that is, if it has some lack of precision, then the corresponding coefficients or elements defining the problem can be formulated by means of fuzzy numbers and thus fuzzy transportation problems arise.

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Several researchers have carried out investigations on fully fuzzy transportation problems with pentagonal and hexagonal fuzzy numbers.

Kaur et al. [1] presented new improved methods for solving the fully fuzzy transportation problems with parameters given as the LR flat fuzzy numbers. Rabinson and Chandrasekaran [5] proposed a method for solving a pentagonal fuzzy transportation problem via ranking technique and ATM. Kalyani and Nagarani [6] proposed a fully fuzzy transportation problem with hexagonal fuzzy number. Thamaraiselvi and Santhi [12] introduced a method to solving fuzzy transportation problem with generalized hexagonal fuzzy numbers. Ghadle and Pathade [13] proposed a model solving transportation problem with generalized hexagonal and generalized octagonal fuzzy numbers by ranking method. Geetha and Selvakumari [14] proposed a new method for solving fuzzy transportation problem using pentagonal fuzzy numbers. Ghadle and Pathade [15] obtained an optimal solution of balanced and unbalanced fuzzy transportation problem using hexagonal fuzzy numbers. Fahrudin and Nuraini [51] proposed performance evaluation of new ranking function methods with current ranking functions using VAM and MM-VAM.

In general, most of the existing techniques provide only crisp solutions for the fully fuzzy transportation problem with pentagonal and hexagonal fuzzy numbers.

In this paper, a new improved method for solving the fully fuzzy transportation problems with parameters given as the pentagonal and hexagonal fuzzy numbers is proposed. Moreover, this new method finds the non-negative fuzzy optimal solution of fully fuzzy transportation problems with parameters given as the pentagonal and hexagonal fuzzy numbers.

The contributions of the present study are summarized as follows: (a) we introduce new technique for improve methods for solving the transportation problems with parameters given as the interval numbers. (b) We introduce a formulation of fully fuzzy transportation problems involving pentagonal and hexagonal fuzzy numbers for the transportation costs and values of supplies and demands. (c) According to the proposed approach, the *Eq. (16)* with pentagonal fuzzy numbers is converted into a classical transportation problem and two interval transportation problems. The integration of the optimal solutions of the three sub-problems provides the optimal solution of the *Eq. (16)*. (d) according to the proposed approach, the *Eq. (16)* with hexagonal fuzzy numbers is converted into three interval transportation problems. The integration of the optimal solutions of the three sub-problems provides the optimal solution of the *Eq. (16)*. (e) an algorithm for the new proposed method and is developed to find optimal solution of the problem. (f) The complexity of computation is greatly reduced compared with commonly used existing methods in the literature.

The rest of this paper is organized as follows. In Section 2, some basic definition, arithmetic operations and interval transportation problems are reviewed. In Section 3, we attempt to introduce a formulation of fully fuzzy transportation problem with pentagonal and hexagonal fuzzy numbers. In Section 4, we propose a simple method for solving fully fuzzy transportation problems and a new fuzzy arithmetic on pentagonal and hexagonal fuzzy numbers. Two application examples are provided to illustrate the effectiveness of the proposed method in Section 5. Advantages of the proposed method over the existing methods are discussed in Section 6. Finally, concluding remarks and future research directions are presented in Section 7.

2 | Materials and Methods

In this section, some basic definitions, arithmetic operations for closed intervals numbers and of linear programming problems involving interval numbers are presented [3] and [4].

2.1 | A New Interval Arithmetic

In this section, some arithmetic operations for two intervals are presented [3] and [4].

Let $\mathfrak{R}=\left\{\bar{a}=\left[a^1,a^2\right]:a^1\le a^2,\ a^1,a^2\in\mathbb{R}\right\}$ be the set of all proper intervals. We shall use the terms "interval" and "interval number" interchangeably. The mid-point and width (or half-width) of an interval number $\bar{a}=\left[a^1,a^2\right]$ are defined as $m(\bar{a})=\frac{a^2+a^1}{2}$ and $w(\bar{a})=\frac{a^2-a^1}{2}$.

The interval number $\bar{a}$ can also be expressed in terms of its midpoint and width as

$$\bar{a}=\left[a^1,a^2\right]=\langle m(\bar{a}),w(\bar{a})\rangle=\langle\frac{a^2+a^1}{2},\frac{a^2-a^1}{2}\rangle. \tag{1}$$

For any two intervals $\bar{a}=\left[a^1,a^2\right]=\langle m(\bar{a}),w(\bar{a})\rangle$ and $\bar{b}=\left[b^1,b^2\right]=\langle m(\bar{b}),w(\bar{b})\rangle$, the arithmetic operations on $\bar{a}$ and $\bar{b}$ are defined as

$$\text{Addition } \bar{a}+\bar{b}=\langle m(\bar{a})+m(\bar{b}),w(\bar{a})+w(\bar{b})\rangle. \tag{2}$$

$$\text{Subtraction } \bar{a}-\bar{b}=\langle m(\bar{a})-m(\bar{b}),w(\bar{a})+w(\bar{b})\rangle. \tag{3}$$

$$\text{Multiplication } \alpha\bar{a}=\begin{cases}\langle\ \alpha m(\bar{a}),\alpha w(\bar{a})\rangle \text{ if } \alpha\ge 0.\\ \langle\ \alpha m(\bar{a}),-\alpha w(\bar{a})\rangle \text{ if } \alpha<0.\end{cases} \tag{4}$$

$$\bar{a}\times\bar{b}=\begin{cases} m(\bar{a})m(\bar{b})+w(\bar{a})w(\bar{b}),m(\bar{a})w(\bar{b})+m(\bar{b})w(\bar{a}) \text{if } a^1\ge 0,b^1\ge 0.\\ m(\bar{a})m(\bar{b})+m(\bar{a})w(\bar{b}),m(\bar{b})w(\bar{a})+w(\bar{b})w(\bar{a}) \text{ if } a^1<0,b^1\ge 0.\\ m(\bar{a})m(\bar{b})-w(\bar{a})w(\bar{b}),m(\bar{b})w(\bar{a})-m(\bar{a})w(\bar{b}) \text{if } a^2<0,b^1\ge 0.\end{cases} \tag{5}$$

2.2 | A New Interval Arithmetic for Pentagonal and Hexagonal Fuzzy Numbers via Intervals Numbers

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

2.2.1 | Pentagonal fuzzy numbers via intervals numbers

A number $\tilde{a}=\left(a^1,a^2,a^3,a^4,a^5\right)$ (where $a^1\le a^2\le a^3\le a^4\le a^5$) is said to be a pentagonal fuzzy number if its membership function $\mu_{\tilde{a}}(x)$ is given by [5] and [7]:

$$\mu_{\tilde{a}}(x)=\begin{cases}0, x<a_1, x\geq a_5,\\ \dfrac{x-a_1}{a_2-a_1}, a_1\leq x\leq a_2,\\ \dfrac{x-a_2}{a_3-a_2}, a_2\leq x\leq a_3,\\ 1, x=a_3,\\ \dfrac{a_4-x}{a_4-a_3}, a_3\leq x\leq a_4,\\ \dfrac{a_5-x}{a_5-a_4}, a_4\leq x\leq a_5.\end{cases} \tag{6}$$

Assume that $\tilde{a}=(a^1,a^2,a^3,a^4,a^5)=(a^3,\overline{a}^{24},\overline{a}^{15})=(a^3,[a^2,a^4],[a^1,a^5])$ where $\overline{a}^{15}=[a^1,a^5]$ and $\overline{a}^{24}=[a^2,a^4]$. For any two pentagonal fuzzy numbers non-negative $\tilde{a}$ and $\tilde{b}$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as

Addition $\tilde{a}+\tilde{b}=(a^3,\overline{a}^{24},\overline{a}^{15})+(b^3,\overline{b}^{24},\overline{b}^{15})=(a^3+b^3,\overline{a}^{24}+\overline{b}^{24},\overline{a}^{15}+\overline{b}^{15}).$ (7)

Multiplication $\tilde{a}+\tilde{b}=(a^3,\overline{a}^{24},\overline{a}^{15})+(b^3,\overline{b}^{24},\overline{b}^{15})=(a^3+b^3,\overline{a}^{24}+\overline{b}^{24},\overline{a}^{15}+\overline{b}^{15}).$ (8)

2.2.2 | Hexagonal fuzzy numbers via intervals numbers

A number $\tilde{a}=(a^1,a^2,a^3,a^4,a^5,a^6)$ (where $a^1\leq a^2\leq a^3\leq a^4\leq a^5\leq a^6$) is said to be a hexagonal fuzzy number if its membership function $\mu_{\tilde{a}}(x)$ is given by [6]:

$$\mu_{\tilde{a}}(x)=\begin{cases}\dfrac{1}{2}\left(\dfrac{x-a_1}{a_2-a_1}\right), a_1\leq x\leq a_2,\\ \dfrac{1}{2}+\dfrac{1}{2}\left(\dfrac{x-a_2}{a_3-a_2}\right), a_2\leq x\leq a_3,\\ 1, a_3\leq x\leq a_4\\ 1-\dfrac{1}{2}\left(\dfrac{x-a_4}{a_5-a_4}\right), a_4\leq x\leq a_5,\\ \dfrac{1}{2}\left(\dfrac{a_6-x}{a_6-a_5}\right), a_5\leq x\leq a_6.\end{cases}$$

Assume that $\tilde{a}=(a^1,a^2,a^3,a^4,a^5,a^6)=(\overline{a}^{34},\overline{a}^{25},\overline{a}^{16})=([a^3,a^4],[a^2,a^5],[a^1,a^6])$ where $\overline{a}^{16}=[a^1,a^6]$, $\overline{a}^{34}=[a^3,a^4]$ and $\overline{a}^{25}=[a^2,a^5]$. For any two hexagonal fuzzy numbers non-negative $\tilde{a}$ and $\tilde{b}$, the arithmetic operations on $\tilde{a}$ and $\tilde{b}$ are defined as:

Addition $\tilde{a}+\tilde{b}=(\overline{a}^{34},\overline{a}^{25},\overline{a}^{16})+(\overline{b}^{34},\overline{b}^{25},\overline{b}^{16})=(\overline{a}^{34}+\overline{b}^{34},\overline{a}^{25}+\overline{b}^{25},\overline{a}^{16}+\overline{b}^{16}).$ (9)

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$$\text{Multiplication}\ \ \tilde{a}\times\tilde{b}=\left(\bar{a}^{34},\bar{a}^{25},\bar{a}^{16}\right)\left(\bar{b}^{34},\bar{b}^{25},\bar{b}^{16}\right)=\left(\bar{a}^{34}\bar{b}^{34},\bar{a}^{25}\bar{b}^{25},\bar{a}^{16}\bar{b}^{16}\right). \tag{10}$$

2.2 | Formulation of a Transportation Problems Involving Interval Numbers

We consider the transportation problem involving interval numbers as follows [3] and [4]:

$$\begin{cases} \text{Min}\,\bar{Z}(\bar{x})\approx\sum_{j=1}^{n}\sum_{i=1}^{m}\bar{c}_{ij}\bar{x}_{ij}, \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n}\bar{x}_{ij}\approx\bar{a}_i,\text{for } i=1,2,..,m, \\ \sum_{i=1}^{m}\bar{x}_{ij}\approx\bar{b}_j,\ \text{for } j=1,2,..,n. \end{cases} \tag{11}$$

where $\bar{c}_{ij}=\left[c_{ij}^1,c_{ij}^2\right]$, $\bar{a}_i=\left[a_i^1,a_i^2\right]$, $\bar{b}_j=\left[b_j^1,b_j^2\right]$ are non-negative interval numbers and $\bar{x}_{ij}=\left[x_{ij}^1,x_{ij}^2\right]$ are unrestricted interval numbers.

Objective function transformation.

$$\bar{Z}(\bar{x})\approx\sum_{j=1}^{n}\sum_{i=1}^{m}\bar{c}_{ij}\bar{x}_{ij}=\sum_{j=1}^{n}\sum_{i=1}^{m}\left[c_{ij}^1,c_{ij}^2\right]\left[x_{ij}^1,x_{ij}^2\right]=\sum_{j=1}^{n}\sum_{i=1}^{m}\langle m\left(\bar{c}_{ij}\bar{x}_{ij}\right),w\left(\bar{c}_{ij}\bar{x}_{ij}\right)\rangle\ \text{ where}$$

$$m\left(\bar{c}_{ij}\bar{x}_{ij}\right)=\begin{cases} m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)+w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)\text{if } x_{ij}^1\geq 0, \\ m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)+w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)\text{if } x_{ij}^1\langle 0 \text{ and } x_{ij}^2\rangle 0, \\ m\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)-w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)\text{if } x_{ij}^2<0. \end{cases} \tag{12}$$

And

$$w\left(\bar{c}_{ij}\bar{x}_{ij}\right)=\begin{cases} m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)+w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)\text{if } x_{ij}^1\geq 0, \\ m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)+w\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)\text{if } x_{ij}^1\langle 0 \text{ and } x_{ij}^2\rangle 0, \\ m\left(\bar{c}_{ij}\right)w\left(\bar{x}_{ij}\right)-w\left(\bar{c}_{ij}\right)m\left(\bar{x}_{ij}\right)\text{if } x_{ij}^2<0. \end{cases} \tag{13}$$

Transformation of constraints.

$$\sum_{j=1}^{n}\bar{x}_{ij}\approx\bar{a}_i\Leftrightarrow\sum_{j=1}^{n}\langle m\left(\bar{x}_{ij}\right),w\left(\bar{x}_{ij}\right)\rangle\approx\langle m\left(\bar{a}_i\right),w\left(\bar{a}_i\right)\rangle\ \text{ for }\ i=1,2,\dots,m\ \text{ and}$$

$$\sum_{i=1}^{m}\bar{x}_{ij}\approx\bar{b}_j\Leftrightarrow\sum_{i=1}^{m}\langle m\left(\bar{x}_{ij}\right),w\left(\bar{x}_{ij}\right)\rangle\approx\langle m\left(\bar{b}_j\right),w\left(\bar{b}_j\right)\rangle\ \text{ for }\ j=1,2,\dots,n.$$

Remark 1. If the decision maker wishes to exhaust supplies:

1) $\sum_{j=1}^{n}\bar{x}_{ij}=\bar{a}_i$ if and only if $\sum_{j=1}^{n}w\left(\bar{x}_{ij}\right)=w\left(\bar{a}_i\right)$ and $\sum_{j=1}^{n}m\left(\bar{x}_{ij}\right)=m\left(\bar{a}_i\right)$ for $i=1,2,\dots,m$.

2) $\sum_{j=1}^{n}\bar{x}_{ij}\neq\bar{a}_i$ if and only if $\sum_{j=1}^{n}w\left(\bar{x}_{ij}\right)\neq w\left(\bar{a}_i\right)$ and $\sum_{j=1}^{n}m\left(\bar{x}_{ij}\right)=m\left(\bar{a}_i\right)$ for $i=1,2,\dots,m$.

Remark 2. If the decision-maker wishes to exhaust the demands:

1) $\sum_{i=1}^{m} \bar{x}_{ij} = \bar{b}_j$ if and only if $\sum_{i=1}^{m} w(\bar{x}_{ij}) = w(\bar{b}_j)$ and $\sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j)$ for $j = 1, 2, \ldots, n$.

2) $\sum_{i=1}^{m} \bar{x}_{ij} \neq \bar{b}_j$ if and only if $\sum_{i=1}^{m} w(\bar{x}_{ij}) \neq w(\bar{b}_j)$ and $\sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j)$ for $j = 1, 2, \ldots, n$.

Now we can say that

$$\begin{cases} \mathrm{Min}\, \bar{Z}(\bar{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m} \langle m(\bar{c}_{ij}\bar{x}_{ij}), w(\bar{c}_{ij}\bar{x}_{ij}) \rangle, \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} \langle m(\bar{x}_{ij}), w(\bar{x}_{ij}) \rangle \approx \langle m(\bar{a}_i), w(\bar{a}_i) \rangle, \text{for } i = 1, \ldots, m, \\ \sum_{i=1}^{m} \langle m(\bar{x}_{ij}), w(\bar{x}_{ij}) \rangle \approx \langle m(\bar{b}_j), w(\bar{b}_j) \rangle, \text{for } j = 1, \ldots, n. \end{cases} \quad (14)$$

is equivalent to

$$\begin{cases} m(\mathrm{Min}\, \bar{Z}(\bar{x})) = \sum_{j=1}^{n}\sum_{i=1}^{m} m(\bar{c}_{ij}\bar{x}_{ij}), \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} m(\bar{x}_{ij}) = m(\bar{a}_i), \text{for } i = 1, 2, .., m, \\ \sum_{i=1}^{m} m(\bar{x}_{ij}) = m(\bar{b}_j), \text{ for } j = 1, 2, .., n. \end{cases} \quad (15)$$

Optimal solution according to the choice of the decision maker:

$\bar{x}_{ij}^{*} = [x_{ij}^{1}, x_{ij}^{2}] = [m(\bar{x}_{ij}) - w(\bar{x}_{ij}), m(\bar{x}_{ij}) + w(\bar{x}_{ij})]$ where $\sum_{j=1}^{n} w(\bar{x}_{ij}) = w(\bar{a}_i)$ and $w(\bar{x}_{ip}) \geq w(\bar{x}_{ik})$ if $c_{ip}^{2} \leq c_{ik}^{2}$ for $i = 1, 2, \ldots, m$.

$\bar{x}_{ij}^{*} = 0$ if and only if $m(\bar{x}_{ij}) = 0$.

3 | Formulation of Fuzzy Transportation Problem with Pentagonal or Hexagonal Fuzzy Numbers

The fuzzy linear programming formulation of a fully fuzzy transportation problem can be written as follows as follows [30] and [31]:

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$$\begin{cases} \operatorname{Min} \tilde{Z}(\tilde{x}) \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}\tilde{x}_{ij}, \\ \text{Subject to the constraints} \\ \sum_{j=1}^{n} \tilde{x}_{ij} \approx \tilde{a}_i, \text{ for } i = 1,2,..,m, \\ \sum_{i=1}^{m} \tilde{x}_{ij} \approx \tilde{b}_j, \text{ for } j = 1,2,..,n. \end{cases} \tag{16}$$

Where $\tilde{x}_{ij}$ are unrestricted pentagonal or hexagonal fuzzy numbers and $\tilde{c}_{ij}$, $\tilde{a}_i$ and $\tilde{b}_j$ are non-negatives pentagonal or hexagonal fuzzy numbers.

3.1 | Transportation Problem with Pentagonal Fuzzy Numbers

For all the rest of this paper, we will consider the following transportation problem with pentagonal fuzzy numbers as follows *Eq.(16)* [5] and [7]:

$\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5\right) = \left(x_{ij}^3, \bar{x}_{ij}^{24}, \bar{x}_{ij}^{15}\right)$ with $\bar{x}_{ij}^{24} = \left[x_{ij}^2, x_{ij}^4\right]$ and $\bar{x}_{ij}^{15} = \left[x_{ij}^1, x_{ij}^5\right]$,

$\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4, c_{ij}^5\right) = \left(c_{ij}^3, \bar{c}_{ij}^{24}, \bar{c}_{ij}^{15}\right)$ with $\bar{c}_{ij}^{24} = \left[c_{ij}^2, c_{ij}^4\right]$ and $\bar{c}_{ij}^{15} = \left[c_{ij}^1, c_{ij}^5\right]$,

$\tilde{a}_i = \left(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5\right) = \left(a_i^3, \bar{a}_i^{24}, \bar{a}_i^{15}\right)$ with $\bar{a}_i^{24} = \left[a_i^2, a_i^4\right]$ and $\bar{a}_i^{15} = \left[a_i^1, a_i^5\right]$ and

$\tilde{b}_j = \left(b_j^1, b_j^2, b_j^3, b_j^4, b_j^5\right) = \left(b_j^3, \bar{b}_j^{24}, \bar{b}_j^{15}\right)$ with $\bar{b}_j^{24} = \left[b_j^2, b_j^4\right]$ and $\bar{b}_j^{15} = \left[b_j^1, b_j^5\right]$.

3.2 | Transportation Problem with Hexagonal Fuzzy Numbers

For all the rest of this paper, we will consider the following transportation problem with hexagonal fuzzy numbers as follows *Eq. (16)* [6]:

$\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6\right) = \left(\bar{x}_{ij}^{34}, \bar{x}_{ij}^{25}, \bar{x}_{ij}^{16}\right)$ with $\bar{x}_{ij}^{34} = \left[x_{ij}^3, x_{ij}^4\right]$, $\bar{x}_{ij}^{25} = \left[x_{ij}^2, x_{ij}^5\right]$ and $\bar{x}_{ij}^{16} = \left[x_{ij}^1, x_{ij}^6\right]$,

$\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4, c_{ij}^5, c_{ij}^6\right) = \left(\bar{c}_{ij}^{34}, \bar{c}_{ij}^{25}, \bar{c}_{ij}^{16}\right)$ with $\bar{c}_{ij}^{34} = \left[c_{ij}^3, c_{ij}^4\right]$, $\bar{c}_{ij}^{25} = \left[c_{ij}^2, c_{ij}^5\right]$ and $\bar{c}_{ij}^{16} = \left[c_{ij}^1, c_{ij}^6\right]$,

$\tilde{a}_i = \left(a_i^1, a_i^2, a_i^3, a_i^4, a_i^5, a_i^6\right) = \left(\bar{a}_i^{34}, \bar{a}_i^{25}, \bar{a}_i^{16}\right)$ with $\bar{a}_i^{34} = \left[a_i^3, a_i^4\right]$, $\bar{a}_i^{25} = \left[a_i^2, a_i^5\right]$ and $\bar{a}_i^{16} = \left[a_i^1, a_i^6\right]$ and

$\tilde{b}_j = \left(b_j^1, b_j^2, b_j^3, b_j^4, b_j^5, b_j^6\right) = \left(\bar{b}_j^{34}, \bar{b}_j^{25}, \bar{b}_j^{16}\right)$ with $\bar{b}_j^{34} = \left[b_j^3, b_j^4\right]$, $\bar{b}_j^{25} = \left[b_j^2, b_j^5\right]$ and $\bar{b}_j^{16} = \left[b_j^1, b_j^6\right]$.

4 | Results

In this section, we will describe our method of solving.

4.1 | Solution Procedure for Transportation Problem with Hexagonal Fuzzy Numbers

Thanks to the new Interval *Arithmetic (9)*, *(10)*, and *(11)*, we can write the following transportation problems involving interval numbers $\bar{x}_{ij}^{34}$, $\bar{c}_{ij}^{34}$, $\bar{a}_i^{34}$ and $\bar{b}_j^{34}$ as follows [3] and [4]:

$$\begin{cases} \operatorname{Min} \overline{Z}^{34}\left(\overline{x}^{34}\right) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{c}_{ij}^{34} \overline{x}_{ij}^{34}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} \overline{x}_{ij}^{34} \approx \overline{a}_{i}^{34}, 1 \le i \le m, \\ \quad \sum_{i=1}^{m} \overline{x}_{ij}^{34} \approx \overline{b}_{j}^{34}, 1 \le j \le n. \end{cases} \tag{17}$$

Thanks to the new interval arithmetic and *Eq. (17)*, we can write the following transportation problem involving midpoint $\overline{x}_{ij}^{34}$, $\overline{c}_{ij}^{34}$, $\overline{a}_{i}^{34}$ and $\overline{b}_{j}^{34}$ as follows [3] and [4]:

$$\begin{cases} \operatorname{Min} Z^{34}\left(x^{34}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} m\left(\overline{c}_{ij}^{34}\right) x_{ij}^{34}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} x_{ij}^{34} = m\left(\overline{a}_{i}^{34}\right), 1 \le i \le m, \\ \quad \sum_{i=1}^{m} x_{ij}^{34} = m\left(\overline{b}_{j}^{34}\right), 1 \le j \le n. \end{cases} \tag{18}$$

Where $x_{ij}^{34} = m\left(\overline{x}_{ij}^{34}\right)$, $m\left(\overline{c}_{ij}^{34}\right) = \dfrac{c_{ij}^{4} + c_{ij}^{3}}{2}$, $m\left(\overline{a}_{i}^{34}\right) = \dfrac{a_{i}^{4} + a_{i}^{3}}{2}$, $m\left(\overline{b}_{j}^{34}\right) = \dfrac{b_{j}^{4} + b_{j}^{3}}{2}$ and $w\left(\overline{a}_{i}^{34}\right) = \dfrac{a_{i}^{4} - a_{i}^{3}}{2}$.

Thanks to the new *Interval Arithmetic (9)*, *(10)*, and *(11)*, we can write the following transportation problems involving interval numbers $\overline{x}_{ij}^{25}$, $\overline{c}_{ij}^{25}$, $\overline{a}_{i}^{25}$ and $\overline{b}_{j}^{25}$ as follows [3] and [4]:

$$\begin{cases} \operatorname{Min} \overline{Z}^{25}\left(\overline{x}^{25}\right) \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \overline{c}_{ij}^{25} \overline{x}_{ij}^{25}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} \overline{x}_{ij}^{25} \approx \overline{a}_{i}^{25}, 1 \le i \le m, \\ \quad \sum_{i=1}^{m} \overline{x}_{ij}^{25} \approx \overline{b}_{j}^{25}, 1 \le j \le n. \end{cases} \tag{19}$$

Thanks to the new interval arithmetic and *Eq. (19)*, we can write the following transportation problem involving Midpoint $\overline{x}_{ij}^{25}$, $\overline{c}_{ij}^{25}$, $\overline{a}_{i}^{25}$ and $\overline{b}_{j}^{25}$ [3] and [4] as follows:

$$\begin{cases} \operatorname{Min} Z^{25}\left(x^{25}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} m\left(\overline{c}_{ij}^{25}\right) x_{ij}^{25}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n} x_{ij}^{25} = m\left(\overline{a}_{i}^{25}\right), 1 \le i \le m, \\ \quad \sum_{i=1}^{m} x_{ij}^{25} = m\left(\overline{b}_{j}^{25}\right), 1 \le j \le n. \end{cases} \tag{20}$$

Where $x_{ij}^{25} = m\left(\overline{x}_{ij}^{25}\right)$, $m\left(\overline{c}_{ij}^{25}\right) = \dfrac{c_{ij}^{5} + c_{ij}^{2}}{2}$, $m\left(\overline{a}_{i}^{25}\right) = \dfrac{a_{i}^{5} + a_{i}^{2}}{2}$, $m\left(\overline{b}_{j}^{25}\right) = \dfrac{b_{j}^{5} + b_{j}^{2}}{2}$ and $w\left(\overline{a}_{i}^{25}\right) = \dfrac{a_{i}^{5} - a_{i}^{2}}{2}$.

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Thanks to the new *Interval Arithmetic (9)*, *(10)*, and *(11)*, we can write the following Transportation Problems involving Interval numbers $\bar{x}_{ij}^{16}$, $\bar{c}_{ij}^{16}$, $\bar{a}_{i}^{16}$ and $\bar{b}_{j}^{16}$ as follows [3] and [4]:

$$\begin{cases} \text{Min}\ \bar{Z}^{16}\left(\bar{x}^{16}\right) \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\bar{c}_{ij}^{16}\bar{x}_{ij}^{16}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n}\bar{x}_{ij}^{16} \approx \bar{a}_{i}^{16}, 1 \le i \le m, \\ \quad \sum_{i=1}^{m}\bar{x}_{ij}^{16} \approx \bar{b}_{j}^{16}, 1 \le j \le n. \end{cases} \tag{21}$$

Thanks to the new interval arithmetic and *Eq. (21)*, we can write the following Transportation Problem involving Midpoint $\bar{x}_{ij}^{16}$, $\bar{c}_{ij}^{16}$, $\bar{a}_{i}^{16}$ and $\bar{b}_{j}^{16}$ [3] and [4] as follows:

$$\begin{cases} \text{Min}Z^{16}\left(x^{16}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n}m\left(\bar{c}_{ij}^{16}\right)x_{ij}^{16}, \\ \quad \text{Subject to the constraints} \\ \quad \sum_{j=1}^{n}x_{ij}^{16} = m\left(\bar{a}_{i}^{16}\right), 1 \le i \le m, \\ \quad \sum_{i=1}^{m}x_{ij}^{16} = m\left(\bar{b}_{j}^{16}\right), 1 \le j \le n. \end{cases} \tag{22}$$

Where $x_{ij}^{16} = m\left(\bar{x}_{ij}^{16}\right)$, $m\left(\bar{c}_{ij}^{16}\right) = \dfrac{c_{ij}^{6} + c_{ij}^{1}}{2}$, $m\left(\bar{a}_{i}^{16}\right) = \dfrac{a_{i}^{6} + a_{i}^{1}}{2}$, $m\left(\bar{b}_{j}^{16}\right) = \dfrac{b_{j}^{6} + b_{j}^{1}}{2}$ and $w\left(\bar{a}_{i}^{16}\right) = \dfrac{a_{i}^{6} - a_{i}^{1}}{2}$.

Thanks to the new interval arithmetic, we can write the following proposition [3] and [4]:

Proposition 1. If $\bar{x}_{ij}^{34} = \left[x_{ij}^{*3}, x_{ij}^{*4}\right]$ is an optimal solution to the *Eq. (17)*, $\bar{x}_{ij}^{25} = \left[x_{ij}^{*2}, x_{ij}^{*5}\right]$ is an optimal solution to the *Eq. (19)* and $\bar{x}_{ij}^{16} = \left[x_{ij}^{*1}, x_{ij}^{*6}\right]$ is an optimal solution to the *Eq. (21)*, then $\tilde{x}^{*} = \left(\tilde{x}_{ij}^{*}\right)_{m\times n}$ is an optimal solution to the *Eq. (16)* with $\tilde{x}_{ij} = \left(x_{ij}^{1}, x_{ij}^{2}, x_{ij}^{3}, x_{ij}^{4}, x_{ij}^{5}, x_{ij}^{6}\right) = \left(\bar{x}_{ij}^{34}, \bar{x}_{ij}^{25}, \bar{x}_{ij}^{16}\right)$.

The steps of our method for solving the fully fuzzy transportation problem involving hexagonal fuzzy numbers as follows:

Step 1. Construct the fully fuzzy transportation problem *Eq. (16)*, and then convert it into a balanced one if it is not.

Step 2. Solving *Eq. (17)* via *Eq. (18)*. For $i = 1, \dots, m$, we have $\bar{x}_{ij}^{34} =$ $\left[x_{ij}^{*3}, x_{ij}^{*4}\right] = \left[x_{ij}^{34} - w\left(\bar{x}_{ij}^{34}\right), x_{ij}^{34} + w\left(\bar{x}_{ij}^{34}\right)\right]$.

Where $\sum_{x_{ij}^{34} \ne 0} w\left(\bar{x}_{ij}^{34}\right) = w\left(\bar{a}_{i}^{34}\right)$ and $w\left(\bar{x}_{ip}^{34}\right) \ge w\left(\bar{x}_{ik}^{34}\right)$ if $c_{ip}^{6} \le c_{ik}^{6}$.

Step 3. Solving *Eq. (19)* via *Eq. (20)*. For $i = 1, \dots, m$: If $E_i = \sum_{j=1}^{n}\left|x_{ij}^{25} - x_{ij}^{34}\right| + w\left(\bar{a}_{i}^{34}\right) \le w\left(\bar{a}_{i}^{25}\right)$, then

$$\bar{x}_{ij}^{25} = \left[x_{ij}^{*2}, x_{ij}^{*5}\right] = \left[x_{ij}^{25} - w\left(\bar{x}_{ij}^{25}\right), x_{ij}^{25} + w\left(\bar{x}_{ij}^{25}\right)\right] \text{ with } \sum_{x_{ij}^{25} \ne 0} w\left(\bar{x}_{ij}^{25}\right) = w\left(\bar{a}_{i}^{25}\right).$$

If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{25} - x_{ij}^{34} \right| + w\left(\bar{a}_i^{34}\right) > w\left(\bar{a}_i^{25}\right)$, then

$\bar{x}_{ij}^{25} = \left[x_{ij}^{*2}, x_{ij}^{*5} \right] = \left[x_{ij}^{25} - w\left(\bar{x}_{ij}^{25}\right), x_{ij}^{25} + w\left(\bar{x}_{ij}^{25}\right) \right]$ with $w\left(\bar{x}_{ij}^{25}\right) = \left| x_{ij}^{25} - x_{ij}^{34} \right| + w\left(\bar{x}_{ij}^{34}\right)$.

Step 4. Solving *Eq. (21)* via *Eq. (22)*. For $i = 1, \ldots, m$: If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{16} - x_{ij}^{25} \right| + w\left(\bar{a}_i^{25}\right) \leq w\left(\bar{a}_i^{16}\right)$, then

$\bar{x}_{ij}^{16} = \left[x_{ij}^{*1}, x_{ij}^{*6} \right] = \left[x_{ij}^{16} - w\left(\bar{x}_{ij}^{16}\right), x_{ij}^{16} + w\left(\bar{x}_{ij}^{16}\right) \right]$ with $\sum_{x_{ij}^{16} \neq 0} w\left(\bar{x}_{ij}^{16}\right) = w\left(\bar{a}_i^{16}\right)$.

If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{16} - x_{ij}^{25} \right| + w\left(\bar{a}_i^{25}\right) > w\left(\bar{a}_i^{16}\right)$, then $\bar{x}_{ij}^{16} = \left[x_{ij}^{*1}, x_{ij}^{*6} \right] = \left[x_{ij}^{16} - w\left(\bar{x}_{ij}^{16}\right), x_{ij}^{16} + w\left(\bar{x}_{ij}^{16}\right) \right]$ with $w\left(\bar{x}_{ij}^{16}\right) = \left| x_{ij}^{16} - x_{ij}^{25} \right| + w\left(\bar{x}_{ij}^{25}\right)$.

Step 5. The optimal solution according to the choice of the decision maker is Min $\tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n} \sum_{i=1}^{m} \tilde{c}_{ij} \tilde{x}_{ij}$ with $\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6 \right) = \left(\bar{x}_{ij}^{34}, \bar{x}_{ij}^{25}, \bar{x}_{ij}^{16} \right)$.

4.2 | Solution Procedure for Transportation Problem with Pentagonal Fuzzy Numbers

The fuzzy number hexagonal $\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6 \right) = \left(\bar{x}_{ij}^{34}, \bar{x}_{ij}^{25}, \bar{x}_{ij}^{16} \right)$ is said to be pentagonal if and only if $x_{ij}^{*3} = x_{ij}^{*4}$. Then $w\left(\bar{x}_{ij}^{34}\right) = \frac{x_{ij}^{*4} - x_{ij}^{*3}}{2} = w\left(\bar{a}_i^{34}\right) = 0$ and $\tilde{x}_{ij} = \left(x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5 \right) = \left(x_{ij}^3, \bar{x}_{ij}^{24}, \bar{x}_{ij}^{15} \right)$. The steps of our method for solving the fully fuzzy transportation problem involving pentagonal fuzzy numbers as follows:

Step 1. Construct the *Fully Fuzzy Transportation Problem (16)*, and then convert it into a balanced one if it is not.

Step 2. Solving *Eq. (17)* via *Eq. (18)*. For $i = 1, \ldots, m$, we have $\bar{x}_{ij}^{34} = \left[x_{ij}^{*3}, x_{ij}^{*4} \right] = \left[x_{ij}^{*3}, x_{ij}^{*3} \right] = x_{ij}^{*3}$.

Step 3. Solving *Eq. (19)* via *Eq. (20)*. For $i = 1, \ldots, m$: If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{24} - x_{ij}^{3} \right| \leq w\left(\bar{a}_i^{24}\right)$, then $\bar{x}_{ij}^{24} = \left[x_{ij}^{*2}, x_{ij}^{*4} \right] = \left[x_{ij}^{24} - w\left(\bar{x}_{ij}^{24}\right), x_{ij}^{24} + w\left(\bar{x}_{ij}^{24}\right) \right]$ with $\sum_{x_{ij}^{24} \neq 0} w\left(\bar{x}_{ij}^{24}\right) = w\left(\bar{a}_i^{24}\right)$.

If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{24} - x_{ij}^{3} \right| > w\left(\bar{a}_i^{24}\right)$, then $\bar{x}_{ij}^{24} = \left[x_{ij}^{*2}, x_{ij}^{*4} \right] = \left[x_{ij}^{24} - w\left(\bar{x}_{ij}^{24}\right), x_{ij}^{24} + w\left(\bar{x}_{ij}^{24}\right) \right]$ with $w\left(\bar{x}_{ij}^{24}\right) = \left| x_{ij}^{24} - x_{ij}^{3} \right|$.

Step 4. Solving *Eq. (21)* via *Eq. (22)*. For $i = 1, \ldots, m$: If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{15} - x_{ij}^{24} \right| + w\left(\bar{a}_i^{24}\right) \leq w\left(\bar{a}_i^{15}\right)$, then

$\bar{x}_{ij}^{15} = \left[x_{ij}^{*1}, x_{ij}^{*5} \right] = \left[x_{ij}^{15} - w\left(\bar{x}_{ij}^{15}\right), x_{ij}^{15} + w\left(\bar{x}_{ij}^{15}\right) \right]$ with $\sum_{x_{ij}^{15} \neq 0} w\left(\bar{x}_{ij}^{15}\right) = w\left(\bar{a}_i^{15}\right)$.

If $E_i = \sum_{j=1}^{n} \left| x_{ij}^{15} - x_{ij}^{24} \right| + w\left(\bar{a}_i^{24}\right) > w\left(\bar{a}_i^{15}\right)$, then

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$$\bar{x}_{ij}^{15}=\left[x_{ij}^{*1},x_{ij}^{*5}\right]=\left[x_{ij}^{15}-w\left(\bar{x}_{ij}^{15}\right),x_{ij}^{15}+w\left(\bar{x}_{ij}^{15}\right)\right] \text{ with } w\left(\bar{x}_{ij}^{15}\right)=\left|x_{ij}^{15}-x_{ij}^{24}\right|+w\left(\bar{x}_{ij}^{24}\right).$$

Step 5. The optimal solution according to the choice of the decision maker is Min $\tilde{Z}(\tilde{x})\approx\sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij}$ with $\tilde{x}_{ij}=\left(x_{ij}^{1},x_{ij}^{2},x_{ij}^{3},x_{ij}^{4},x_{ij}^{5}\right)=\left(x_{ij}^{3},\bar{x}_{ij}^{24},\bar{x}_{ij}^{15}\right)$.

5 | Numerical Examples

The proposed method is explained through the illustrations.

Example 1. Consider hexagonal fuzzy transportation problem with three sources that is S_1, S_2, S_3 and three destinations D_1, D_2, D_3 shown in *Table 1*.

Table 1. Hexagonal fuzzy transportation table.

	D_1	D_2	D_3	Supply
S_1	(3,7,11,15,19,24)	(3,5,7,9,10,12)	(11,14,17,21,25,30)	(7,9,11,13,16,20)
S_2	(3,5,7,9,10,12)	(5,7,10,13,17,21)	(7,9,11,14,18,22)	(6,8,11,14,19,25)
S_3	(7,9,11,14,18,22)	(2,3,4,6,7,9)	(5,7,8,11,14,17)	(9,11,13,15,18,20)
Demand	(6,9,12,15,20,25)	(6,7,9,11,13,16)	(10,12,14,16,20,24)	$\sum_{i=1}^{m}\tilde{a}_i=\sum_{j=1}^{n}\tilde{b}_j$

Step 1. Construct the *Fully Fuzzy Transportation Problem (16)*, and then convert it into a balanced one if it is not.

Step 2. Solving *Eq. (17)* via *Eq. (18)*. We have $m\left(\bar{c}_{ij}^{34}\right)=\frac{c_{ij}^{4}+c_{ij}^{3}}{2}$, $m\left(\bar{a}_{i}^{34}\right)=\frac{a_{i}^{4}+a_{i}^{3}}{2}$ and $m\left(\bar{b}_{j}^{34}\right)=\frac{b_{j}^{4}+b_{j}^{3}}{2}$ and $w\left(\bar{a}_{i}^{34}\right)=\frac{a_{i}^{4}-a_{i}^{3}}{2}$.

$$MinZ^{34}\left(x^{34}\right)=\frac{26}{2}x_{11}^{34}+\frac{16}{2}x_{12}^{34}+\frac{38}{2}x_{13}^{34}+\frac{16}{2}x_{21}^{34}+\frac{23}{2}x_{22}^{34}+\frac{25}{2}x_{23}^{34}+\frac{25}{2}x_{31}^{34}+\frac{10}{2}x_{32}^{34}+\frac{19}{2}x_{33}^{34},$$

Subject to the constraints $x_{11}^{34}+x_{12}^{34}+x_{13}^{34}=\frac{24}{2}$, $x_{21}^{34}+x_{22}^{34}+x_{23}^{34}=\frac{25}{2}$, $x_{31}^{34}+x_{32}^{34}+x_{33}^{34}=\frac{28}{2}$, $x_{11}^{34}+x_{21}^{34}+x_{31}^{34}=\frac{27}{2}$, $x_{12}^{34}+x_{22}^{34}+x_{32}^{34}=\frac{20}{2}$, $x_{13}^{34}+x_{23}^{34}+x_{33}^{34}=\frac{30}{2}$.

Optimal solution. $x_{11}^{34}=2,x_{12}^{34}=10,x_{13}^{34}=0,x_{21}^{34}=\frac{23}{2},x_{22}^{34}=0,x_{23}^{34}=1,x_{31}^{34}=0,x_{32}^{34}=0$ and $x_{33}^{34}=14$.

Furthermore $w\left(\bar{a}_{1}^{34}\right)=\frac{2}{2}=1$, $w\left(\bar{a}_{2}^{34}\right)=\frac{3}{2}$, $w\left(\bar{a}_{3}^{34}\right)=\frac{2}{2}=1$.

For $i=1$, then $\sum_{x_{1j}^{34}\neq 0}w\left(\bar{x}_{1j}^{34}\right)=w\left(\bar{a}_{1}^{34}\right)=1$ with $w\left(\bar{x}_{11}^{34}\right)=\frac{1}{4}$ and $w\left(\bar{x}_{12}^{34}\right)=\frac{3}{4}$.

We get $\bar{x}_{11}^{34}=\left[\frac{7}{4},\frac{9}{4}\right]$, $\bar{x}_{12}^{34}=\left[\frac{37}{4},\frac{43}{4}\right]$ and $\bar{x}_{13}^{34}=[0,0]$.

For $i=2$, then $\sum_{x_{2j}^{34}\neq 0} w\left(\overline{x}_{2j}^{34}\right)=w\left(\overline{a}_2^{34}\right)=\frac{3}{2}$ with $w\left(\overline{x}_{21}^{34}\right)=1$ and $w\left(\overline{x}_{23}^{34}\right)=\frac{1}{2}$.

We get $\overline{x}_{21}^{34}=\left[\frac{21}{2},\frac{25}{2}\right]$, $\overline{x}_{22}^{34}=[0,0]$ and $\overline{x}_{23}^{34}=\left[\frac{1}{2},\frac{3}{2}\right]$.

For $i=3$, then $\sum_{x_{3j}^{34}\neq 0} w\left(\overline{x}_{3j}^{34}\right)=w\left(\overline{a}_3^{34}\right)=1$ with $w\left(\overline{x}_{33}^{34}\right)=1$.

We get $\overline{x}_{31}^{34}=[0,0]$, $\overline{x}_{32}^{34}=[0,0]$ and $\overline{x}_{33}^{34}=[13,15]$.

Step 3. Solving *Eq. (19)* via *Eq. (20)*. We have $m\left(\overline{c}_{ij}^{25}\right)=\frac{c_{ij}^5+c_{ij}^2}{2}$, $m\left(\overline{a}_i^{25}\right)=\frac{a_i^5+a_i^2}{2}$ and $m\left(\overline{b}_j^{25}\right)=\frac{b_j^5+b_j^2}{2}$ and $w\left(\overline{a}_i^{25}\right)=\frac{a_i^5-a_i^2}{2}$.

$$Min\,Z^{25}\left(x^{25}\right)=\frac{26}{2}x_{11}^{25}+\frac{15}{2}x_{12}^{25}+\frac{39}{2}x_{13}^{25}+\frac{15}{2}x_{21}^{25}+\frac{24}{2}x_{22}^{25}+\frac{27}{2}x_{23}^{25}+\frac{27}{2}x_{31}^{25}+\frac{10}{2}x_{32}^{25}+\frac{21}{2}x_{33}^{25},$$

Subject to the constraints $x_{11}^{25}+x_{12}^{25}+x_{13}^{25}=\frac{25}{2}$, $x_{21}^{25}+x_{22}^{25}+x_{23}^{25}=\frac{27}{2}$, $x_{31}^{25}+x_{32}^{25}+x_{33}^{25}=\frac{29}{2}$, $x_{11}^{25}+x_{21}^{25}+x_{31}^{25}=\frac{29}{2}$, $x_{12}^{25}+x_{22}^{25}+x_{32}^{25}=\frac{20}{2}$, $x_{13}^{25}+x_{23}^{25}+x_{33}^{25}=\frac{32}{2}$.

Optimal solution. $x_{11}^{25}=\frac{5}{2}, x_{12}^{25}=10, x_{13}^{25}=0, x_{21}^{25}=12, x_{22}^{25}=0, x_{23}^{25}=\frac{3}{2}$, $x_{31}^{25}=0, x_{32}^{25}=0$ and $x_{33}^{25}=\frac{29}{2}$.

Furthermore $w\left(\overline{a}_1^{25}\right)=\frac{7}{2}$, $w\left(\overline{a}_2^{25}\right)=\frac{11}{2}$, $w\left(\overline{a}_3^{25}\right)=\frac{7}{2}$.

For $i=1$, then $E_1=\sum_{j=1}^{3}\left|x_{1j}^{25}-x_{1j}^{34}\right|+w\left(\overline{a}_1^{34}\right)=\left|\frac{5}{2}-2\right|+|10-10|+|0-0|+1=\frac{3}{2}\leq w\left(\overline{a}_1^{25}\right)=\frac{7}{2}$.

We have $\sum_{x_{1j}^{25}\neq 0} w\left(\overline{x}_{1j}^{25}\right)=w\left(\overline{a}_1^{25}\right)=\frac{7}{2}$ with $w\left(\overline{x}_{11}^{25}\right)=\frac{3}{2}$ and $w\left(\overline{x}_{12}^{25}\right)=2$.

We get $\overline{x}_{11}^{25}=[1,4]$, $\overline{x}_{12}^{25}=[8,12]$ and $\overline{x}_{13}^{25}=[0,0]$.

For $i=2$, then $E_2=\sum_{j=1}^{3}\left|x_{2j}^{25}-x_{2j}^{34}\right|+w\left(\overline{a}_2^{34}\right)=\left|12-\frac{23}{2}\right|+|0-0|+\left|\frac{3}{2}-1\right|+\frac{3}{2}=\frac{5}{2}\leq w\left(\overline{a}_2^{25}\right)=\frac{11}{2}$.

We have $\sum_{x_{2j}^{25}\neq 0} w\left(\overline{x}_{2j}^{25}\right)=w\left(\overline{a}_2^{25}\right)$ with $w\left(\overline{x}_{21}^{25}\right)=\frac{9}{2}$ and $w\left(\overline{x}_{23}^{25}\right)=1$.

We get $\overline{x}_{21}^{25}=\left[\frac{15}{2},\frac{33}{2}\right]$, $\overline{x}_{22}^{25}=[0,0]$ and $\overline{x}_{23}^{25}=\left[\frac{1}{2},\frac{5}{2}\right]$.

For $i=3$, then $E_3=\sum_{j=1}^{3}\left|x_{3j}^{25}-x_{3j}^{34}\right|+w\left(\overline{a}_3^{34}\right)=|0-0|+|0-0|+\left|\frac{29}{2}-14\right|+1=\frac{3}{2}\leq w\left(\overline{a}_3^{25}\right)=\frac{7}{2}$.

We have $\sum_{x_{3j}^{25}\neq 0} w\left(\bar{x}_{3j}^{25}\right)=w\left(\bar{a}_3^{25}\right)$ with $w\left(\bar{x}_{33}^{25}\right)=\frac{7}{2}$.

We get $\bar{x}_{31}^{25}=[0,0]$, $\bar{x}_{32}^{25}=[0,0]$ and $\bar{x}_{33}^{25}=[11,18]$.

Step 4. Solving *Eq. (21)* via *Eq. (22).* We have $m\left(\bar{c}_{ij}^{16}\right)=\frac{c_{ij}^6+c_{ij}^1}{2}$, $m\left(\bar{a}_i^{16}\right)=\frac{a_i^6+a_i^1}{2}$,

$m\left(\bar{b}_j^{16}\right)=\frac{b_j^6+b_j^1}{2}$ and $w\left(\bar{a}_i^{16}\right)=\frac{a_i^6-a_i^1}{2}$.

$$Min\, Z^{16}\left(x^{16}\right)=\frac{27}{2}x_{11}^{16}+\frac{15}{2}x_{12}^{16}+\frac{41}{2}x_{13}^{16}+\frac{15}{2}x_{21}^{16}+\frac{26}{2}x_{22}^{16}+\frac{29}{2}x_{23}^{16}+\frac{29}{2}x_{31}^{16}+\frac{11}{2}x_{32}^{16}+\frac{22}{2}x_{33}^{16},$$

Subject to the constraints $x_{11}^{16}+x_{12}^{16}+x_{13}^{16}=\frac{27}{2}$, $x_{21}^{16}+x_{22}^{16}+x_{23}^{16}=\frac{31}{2}$,

$x_{31}^{16}+x_{32}^{16}+x_{33}^{16}=\frac{29}{2}$, $x_{11}^{16}+x_{21}^{16}+x_{31}^{16}=\frac{31}{2}$, $x_{12}^{16}+x_{22}^{16}+x_{32}^{16}=\frac{22}{2}$, $x_{13}^{16}+x_{23}^{16}+x_{33}^{16}=\frac{34}{2}$.

Optimal solution. $x_{11}^{16}=0, x_{12}^{16}=11, x_{13}^{16}=\frac{5}{2}, x_{21}^{16}=\frac{31}{2}, x_{22}^{16}=0, x_{23}^{16}=0,$ $x_{31}^{16}=0, x_{32}^{16}=0$ and $x_{33}^{16}=\frac{29}{2}$.
Furthermore $w\left(\bar{a}_1^{16}\right)=\frac{13}{2}$, $w\left(\bar{a}_2^{16}\right)=\frac{19}{2}$, $w\left(\bar{a}_3^{16}\right)=\frac{11}{2}$.

For $i=1$, then $E_1=\sum_{j=1}^{3}\left|x_{1j}^{16}-x_{1j}^{25}\right|+w\left(\bar{a}_1^{25}\right)=\left|0-\frac{5}{2}\right|+\left|11-10\right|+\left|\frac{5}{2}-0\right|+\frac{7}{2}=\frac{19}{2}>w\left(\bar{a}_1^{16}\right)=\frac{13}{2}$, We have $\sum_{x_{1j}^{16}\neq 0} w\left(\bar{x}_{1j}^{16}\right)>w\left(\bar{a}_1^{16}\right)=\frac{13}{2}$ with $w\left(\bar{x}_{11}^{16}\right)=4$, $w\left(\bar{x}_{12}^{16}\right)=3$ and $w\left(\bar{x}_{13}^{16}\right)=\frac{5}{2}$.

We get $\bar{x}_{11}^{16}=[-4,4]$, $\bar{x}_{12}^{16}=[8,14]$ and $\bar{x}_{13}^{16}=[0,5]$.

For $i=2$, then

$$E_2=\sum_{j=1}^{3}\left|x_{2j}^{16}-x_{2j}^{25}\right|+w\left(\bar{a}_2^{25}\right)=\left|\frac{31}{2}-12\right|+\left|0-0\right|+\left|0-\frac{3}{2}\right|+\frac{11}{2}=\frac{21}{2}>w\left(\bar{a}_2^{15}\right)=\frac{19}{2}.$$

We have $\sum_{x_{2j}^{16}\neq 0} w\left(\bar{x}_{2j}^{16}\right)>w\left(\bar{a}_2^{16}\right)=\frac{19}{2}$ with $w\left(\bar{x}_{21}^{16}\right)=8$, $w\left(\bar{x}_{22}^{16}\right)=0$ and $w\left(\bar{x}_{23}^{16}\right)=\frac{5}{2}$.

We get $\bar{x}_{21}^{16}=\left[\frac{15}{2},\frac{47}{2}\right]$, $\bar{x}_{22}^{16}=[0,0]$ and $\bar{x}_{23}^{16}=\left[\frac{-5}{2},\frac{5}{2}\right]$.

For $i=3$, then $E_3=\sum_{j=1}^{3}\left|x_{3j}^{16}-x_{3j}^{25}\right|+w\left(\bar{a}_3^{25}\right)=\left|0-0\right|+\left|0-0\right|+\left|\frac{29}{2}-\frac{29}{2}\right|+\frac{7}{2}=\frac{7}{2}\leq w\left(\bar{a}_3^{16}\right)=\frac{11}{2}$. We have $\sum_{x_{3j}^{16}\neq 0} w\left(\bar{x}_{3j}^{16}\right)=w\left(\bar{a}_3^{16}\right)=\frac{11}{2}$ with $w\left(\bar{x}_{33}^{16}\right)=\frac{11}{2}$.

We get $\bar{x}_{31}^{16}=[0,0]$, $\bar{x}_{32}^{16}=[0,0]$ and $\bar{x}_{33}^{16}=[9,20]$.

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Step 5. The optimal solution according to the choice of the decision maker is

$$\text{Min} \quad \tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij} \text{ with } \tilde{x}_{ij} = \left(x^1_{ij}, x^2_{ij}, x^3_{ij}, x^4_{ij}, x^5_{ij}, x^6_{ij}\right) = \left(\bar{x}^{34}_{ij}, \bar{x}^{25}_{ij}, \bar{x}^{16}_{ij}\right).$$

$$\tilde{x}_{11} = \left(-4, 1, \frac{7}{4}, \frac{9}{4}, 4, 4\right),\ \tilde{x}_{12} = \left(8, 8, \frac{37}{4}, \frac{43}{4}, 12, 14\right),\ \tilde{x}_{13} = (0,0,0,0,0,5),$$

$$\tilde{x}_{21} = \left(\frac{15}{2}, \frac{15}{2}, \frac{21}{2}, \frac{25}{2}, \frac{33}{2}, \frac{47}{2}\right),\ \tilde{x}_{22} = \tilde{0},\ \tilde{x}_{23} = \left(\frac{-5}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right),\ \tilde{x}_{31} = \tilde{0},\ \tilde{x}_{32} = \tilde{0} \text{ and}$$

$$\tilde{x}_{33} = (9, 11, 13, 15, 18, 20) \text{ with Min } \tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij} = (62, 166, 267, 429, 658, 1091).$$

Example 2. Consider pentagonal fuzzy transportation problem with three sources that is S_1, S_2, S_3 and three destinations D_1, D_2, D_3 shown in *Table 2*.

Table 2. Pentagonal fuzzy transportation table.

	D_1	D_2	D_3	Supply
S_1	$(1,3,4,5,7)$	$(0,2,3,4,6)$	$(2,4,5,6,8)$	$(4,6,7,8,10)$
S_2	$(3,5,6,7,9)$	$(5,7,8,9,11)$	$(4,6,7,8,10)$	$(6,8,9,10,12)$
S_3	$(1,3,4,5,7)$	$(2,4,5,6,8)$	$(3,5,6,7,9)$	$(10,12,13,14,16)$
Demand	$(3,5,6,7,9)$	$(5,7,8,9,11)$	$(12,14,15,16,18)$	$\sum_{i=1}^{m}\tilde{a}_i = \sum_{j=1}^{n}\tilde{b}_j$

Step 1. Construct the *Fully Fuzzy Transportation Problem (16)*, and then convert it into a balanced one if it is not.

Step 2. Solving *Eq. (17)* via *Eq. (18)*.

$$Min\ Z^3\left(x^3\right) = 4x^3_{11} + 3x^3_{12} + 5x^3_{13} + 6x^3_{21} + 8x^3_{22} + 7x^3_{23} + 4x^3_{31} + 5x^3_{32} + 6x^3_{33},$$

Subject to the constraints $x^3_{11} + x^3_{12} + x^3_{13} = 7$, $x^3_{21} + x^3_{22} + x^3_{23} = 9$, $x^3_{31} + x^3_{32} + x^3_{33} = 13$, $x^3_{11} + x^3_{21} + x^3_{31} = 6$, $x^3_{12} + x^3_{22} + x^3_{32} = 8$, $x^3_{13} + x^3_{23} + x^3_{33} = 15$.

Optimal solution. $x^3_{11} = 0, x^3_{12} = 7, x^3_{13} = 0,\ x^3_{21} = 0, x^3_{22} = 0, x^3_{23} = 9,\ x^3_{31} = 6, x^3_{32} = 1$ and $x^3_{33} = 6$.

Step 3. Solving *Eq. (19)* via *Eq. (20)*. We have $m\left(\bar{c}^{24}_{ij}\right) = \frac{c^4_{ij} + c^2_{ij}}{2}$, $m\left(\bar{a}^{24}_{i}\right) = \frac{a^4_i + a^2_i}{2}$ and $m\left(\bar{b}^{24}_{j}\right) = \frac{b^4_j + b^2_j}{2}$ and $w\left(\bar{a}^{24}_{i}\right) = \frac{a^4_i - a^2_i}{2}$.

$$MinZ^{24}\left(x^{24}\right) = \frac{8}{2}x^{24}_{11} + \frac{6}{2}x^{24}_{12} + \frac{10}{2}x^{24}_{13} + \frac{12}{2}x^{24}_{21} + \frac{16}{2}x^{24}_{22} + \frac{14}{2}x^{24}_{23} + \frac{8}{2}x^{24}_{31} + \frac{10}{2}x^{24}_{32} + \frac{12}{2}x^{24}_{33},$$

Subject to the constraints $x^{24}_{11} + x^{24}_{12} + x^{24}_{13} = \frac{14}{2}$, $x^{24}_{21} + x^{24}_{22} + x^{24}_{23} = \frac{18}{2}$, $x^{24}_{31} + x^{24}_{32} + x^{24}_{33} = \frac{26}{2}$, $x^{24}_{11} + x^{24}_{21} + x^{24}_{31} = \frac{12}{2}$, $x^{24}_{12} + x^{24}_{22} + x^{24}_{32} = \frac{16}{2}$, $x^{24}_{13} + x^{24}_{23} + x^{24}_{33} = \frac{30}{2}$.

Optimal solution. $x_{11}^{24}=0, x_{12}^{24}=7, x_{13}^{24}=0, x_{21}^{24}=0, x_{22}^{24}=0, x_{23}^{24}=9,\ \ x_{31}^{24}=6, x_{32}^{24}=1$ and $x_{33}^{24}=6$. Furthermore $w\left(\bar{a}_1^{24}\right)=\frac{2}{2}=1$, $w\left(\bar{a}_2^{24}\right)=\frac{2}{2}=1$, $w\left(\bar{a}_3^{24}\right)=\frac{2}{2}=1$.

For $i=1$, then $E_1=\sum_{j=1}^{3}\left|x_{1j}^{24}-x_{1j}^{3}\right|=|0-0|+|7-7|+|0-0|=0\le w\left(\bar{a}_1^{24}\right)=1$.

We have $\sum_{x_{1j}^{24}\neq 0} w\left(\bar{x}_{1j}^{24}\right)=w\left(\bar{a}_1^{24}\right)=1$ with $w\left(\bar{x}_{22}^{24}\right)=1$.

We get $\bar{x}_{11}^{24}=[0,0]$, $\bar{x}_{12}^{24}=[6,8]$ and $\bar{x}_{13}^{24}=[0,0]$.

For $i=2$, then $E_2=\sum_{j=1}^{3}\left|x_{2j}^{24}-x_{2j}^{3}\right|=|0-0|+|0-0|+|9-9|=0\le w\left(\bar{a}_2^{24}\right)=1$.

We have $\sum_{x_{2j}^{24}\neq 0} w\left(\bar{x}_{2j}^{24}\right)=w\left(\bar{a}_2^{24}\right)=1$ with $w\left(\bar{x}_{23}^{24}\right)=1$.

We get $\bar{x}_{21}^{24}=[0,0]$, $\bar{x}_{22}^{24}=[0,0]$ and $\bar{x}_{23}^{24}=[8,10]$.

For $i=3$, then $E_3=\sum_{j=1}^{3}\left|x_{3j}^{24}-x_{3j}^{3}\right|=|6-6|+|1-1|+|6-6|=0\le w\left(\bar{a}_3^{24}\right)=1$.

We have $\sum_{x_{3j}^{24}\neq 0} w\left(\bar{x}_{3j}^{24}\right)=w\left(\bar{a}_3^{24}\right)=1$ with $w\left(\bar{x}_{31}^{24}\right)=w\left(\bar{x}_{32}^{24}\right)=w\left(\bar{x}_{33}^{24}\right)=\frac{1}{3}$.

We get $\bar{x}_{31}^{24}=\left[\frac{17}{3},\frac{19}{3}\right]$, $\bar{x}_{32}^{24}=\left[\frac{2}{3},\frac{4}{3}\right]$ and $\bar{x}_{33}^{24}=\left[\frac{17}{3},\frac{19}{3}\right]$.

Step 4. Solving *Eq. (21)* via *Eq. (22)*. We have $m\left(\bar{c}_{ij}^{15}\right)=\frac{c_{ij}^{5}+c_{ij}^{1}}{2}$, $m\left(\bar{a}_i^{15}\right)=\frac{a_i^{5}+a_i^{1}}{2}$ and $m\left(\bar{b}_j^{15}\right)=\frac{b_j^{5}+b_j^{1}}{2}$.

$$Min\ Z^{15}\left(x^{15}\right)=\frac{8}{2}x_{11}^{15}+\frac{6}{2}x_{12}^{15}+\frac{10}{2}x_{13}^{15}+\frac{12}{2}x_{21}^{15}+\frac{16}{2}x_{22}^{15}+\frac{14}{2}x_{23}^{15}+\frac{8}{2}x_{31}^{15}+\frac{10}{2}x_{32}^{15}+\frac{12}{2}x_{33}^{15},$$

Subject to the constraints $x_{11}^{15}+x_{12}^{15}+x_{13}^{15}=\frac{14}{2}$, $x_{21}^{15}+x_{22}^{15}+x_{23}^{15}=\frac{18}{2}$, $x_{31}^{15}+x_{32}^{15}+x_{33}^{15}=\frac{26}{2}$, $x_{11}^{15}+x_{21}^{15}+x_{31}^{15}=\frac{12}{2}$, $x_{12}^{15}+x_{22}^{15}+x_{32}^{15}=\frac{16}{2}$, $x_{13}^{15}+x_{23}^{15}+x_{33}^{15}=\frac{30}{2}$.

Optimal solution. $x_{11}^{15}=0, x_{12}^{15}=7, x_{13}^{15}=0, x_{21}^{15}=0, x_{22}^{15}=0, x_{23}^{15}=9,\ \ x_{31}^{15}=6, x_{32}^{15}=1$ and $x_{33}^{15}=6$.

Furthermore $w\left(\bar{a}_1^{15}\right)=\frac{6}{2}=3$, $w\left(\bar{a}_2^{15}\right)=\frac{6}{2}=3$, $w\left(\bar{a}_3^{15}\right)=\frac{6}{2}=3$.

For $i=1$, then $E_1=\sum_{j=1}^{3}\left|x_{1j}^{15}-x_{1j}^{24}\right|+w\left(\bar{a}_1^{24}\right)=|0-0|+|7-7|+|0-0|+1=1\le w\left(\bar{a}_1^{15}\right)=3$.

We have $\sum_{x_{1j}^{15}\neq 0} w\left(\bar{x}_{1j}^{15}\right)=w\left(\bar{a}_1^{15}\right)=3$ with $w\left(\bar{x}_{12}^{15}\right)=3$.

We get $\bar{x}_{11}^{15}=[0,0]$, $\bar{x}_{12}^{15}=[4,10]$ and $\bar{x}_{13}^{15}=[0,0]$.

For $i=2$, then $E_2=\sum_{j=1}^{3}\left|x_{2j}^{15}-x_{2j}^{24}\right|+w\left(\bar{a}_2^{24}\right)=|0-0|+|0-0|+|9-9|+1=1\le w\left(\bar{a}_2^{15}\right)=3$.

We have $\sum_{x_{2j}^{15}\neq 0} w\left(\bar{x}_{2j}^{15}\right)=w\left(\bar{a}_2^{15}\right)=3$ with $w\left(\bar{x}_{23}^{15}\right)=3$.

We get $\bar{x}_{21}^{15}=[0,0]$, $\bar{x}_{22}^{15}=[0,0]$ and $\bar{x}_{23}^{15}=[6,12]$.

For $i=3$, then $E_3=\sum_{j=1}^{3}\left|x_{3j}^{15}-x_{3j}^{24}\right|+w\left(\bar{a}_3^{24}\right)=|6-6|+|1-1|+|6-6|+1=1\le w\left(\bar{a}_3^{15}\right)=3$.

We have $\sum_{x_{3j}^{15}\neq 0} w\left(\bar{x}_{3j}^{15}\right)=w\left(\bar{a}_3^{15}\right)=3$ with $w\left(\bar{x}_{31}^{15}\right)=\frac{3}{2}, w\left(\bar{x}_{32}^{15}\right)=1, w\left(\bar{x}_{33}^{15}\right)=\frac{1}{2}$.

We get $\bar{x}_{31}^{15}=\left[\frac{9}{2},\frac{15}{2}\right]$, $\bar{x}_{32}^{15}=[0,2]$ and $\bar{x}_{33}^{15}=\left[\frac{11}{2},\frac{13}{2}\right]$.

Step 5. The optimal solution according to the choice of the decision maker is Min $\tilde{Z}(\tilde{x})\approx\sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij}$ with $\tilde{x}_{ij}=\left(x_{ij}^{3},\bar{x}_{ij}^{24},\bar{x}_{ij}^{15}\right)=\left(x_{ij}^{1},x_{ij}^{2},x_{ij}^{3},x_{ij}^{4},x_{ij}^{5}\right)$.

$\tilde{x}_{11}=\tilde{0}$, $\tilde{x}_{12}=(4,6,7,8,10)$, $\tilde{x}_{13}=\tilde{0}$, $\tilde{x}_{21}=\tilde{0}$, $\tilde{x}_{22}=\tilde{0}$, $\tilde{x}_{23}=(6,8,9,10,12)$,

$\tilde{x}_{31}=\left(\frac{9}{2},\frac{17}{3},6,\frac{19}{3},\frac{15}{2}\right)$, $\tilde{x}_{32}=\left(0,\frac{2}{3},1,\frac{4}{3},2\right)$ and $\tilde{x}_{33}=\left(\frac{11}{2},\frac{17}{3},6,\frac{19}{3},\frac{13}{2}\right)$ with Min

$\tilde{Z}(\tilde{x})\approx\sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij}=(45,108,149,196,307)$.

6 | Advantages of the Proposed Method over the Existing Methods

To be more specific, we will concentrate on showing the advantages of the proposed method over the well-known existing methods existing methods proposed by [1], [5], [6], [8], [9], [10], [12], [13], and [14]-[16].

The advantages of the new method proposed over the existing methods proposed by [1], [5], [6], [8], [9], [10], [12], [13], and [14]-[16] can be summarized as follows:

The new algorithm improves the existing methods for solving the fully fuzzy transportation problems with parameters given as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, pentagonal fuzzy numbers and hexagonal fuzzy numbers [11].

The proposed technique does not use the goal and parametric approaches which are difficult to apply in real life situations. These difficulties (or limitations) are overcome by the new proposed method.

The proposed method to solve *Eq. (16)* is based on traditional transportation algorithms. Thus, the existing and easily available software can be used for the same. However, the existing method to solve *Eq.(16)* should be implemented into a programming language.

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To solve the *Eq.(16)* by using the existing method, there is need to use arithmetic operations of generalized fuzzy numbers. While, if the proposed technique is used for the same then there is need to use arithmetic operations of real numbers. This proves that it is much easy to apply the proposed method as compared to the existing method.

In contrast to most existing approaches, which provide an optimal solution using ranking function, the proposed method provides a fuzzy optimal solution without using ranking function. Similarly, to the competing methods in the literature, the proposed method is applicable for all types of pentagonal and hexagonal fuzzy numbers.

Moreover, it has also been pointed out that fuzzy optimal solution, obtained by using the existing methods, does not satisfy the constraints exactly. Since in the new method proposed all the cost parameters are represented by pentagonal or hexagonal fuzzy numbers and also the constraint should hold, then while solving the fully fuzzy transportation problems by using these new method the obtained minimum total fuzzy transportation cost and fuzzy quantity of the product will be either zero or non-negative. Also, the fuzzy optimal solution, obtained by using the new method mentioned, will always exactly satisfy the centers of all the constraints and the supplies (or demands) constraints are all almost exhausted.

Our new method finds the non-negative fuzzy optimal solution of balanced and unbalanced fully fuzzy transport problems with parameters given as interval numbers, triangular fuzzy numbers, trapezoidal fuzzy numbers, pentagonal fuzzy numbers and hexagonal fuzzy numbers.

7 | Concluding Remarks and Future Research Directions

7.1 | Concluding Remarks

These days a number of researchers have shown interest in the area of fuzzy transportation problems and various attempts have been made to study the solution of these problems. In this paper, to overcome the shortcomings of the existing methods we introduced a new formulation of transportation problem involving pentagonal or hexagonal fuzzy numbers for the transportation costs and values of supplies and demands. We propose a fuzzy linear programming approach for solving pentagonal or hexagonal fuzzy numbers transportation problem based on the converting into interval transportation problems. To show the advantages of the proposed methods over existing methods, some fuzzy transportation problems, may or may not be solved by the existing methods, are solved by using the proposed methods and it is shown that it is better to use the proposed methods as compared to the existing methods for solving the transportation problems. From both theoretical and algorithmic considerations, and examples solved in this paper, it can be noticed that some shortcomings of the methods for solving the fuzzy transportation problems known from the literature can be resolved by using the new methods proposed in Sect. 4. With the new method, the centers of all constraints are satisfied and the supplies (or demands) are almost exhausted unlike the existing methods.

7.2 | Future Research Directions

Finally, we feel that, there are many other points of research and should be studied later on Interval numbers or Fuzzy numbers. Some of these points are below:

We will consider the following transportation problems *Eq. (16)* with fuzzy numbers as follows:

$Min\,\tilde{Z}(\tilde{x}) \approx \sum_{j=1}^{n}\sum_{i=1}^{m}\tilde{c}_{ij}\tilde{x}_{ij}$ subject to the constraints $\sum_{j=1}^{n}\tilde{x}_{ij} \approx \tilde{a}_i$ and $\sum_{i=1}^{m}\tilde{x}_{ij} \approx \tilde{b}_j$ where

$\tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, \ldots, c_{ij}^t)$, $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^t)$, $\tilde{a}_i = (a_i^1, a_i^2, \ldots, a_i^t)$ and $\tilde{b}_j = (b_j^1, b_j^2, \ldots, b_j^t)$ with $t \in \mathbb{N}_{\geq 1}$. Let $\overline{x}_{ij}^{pq} = [x_{ij}^p, x_{ij}^q]$ where $p \leq q$ and $p, q \in \mathbb{N}_{\geq 1}$. The same applies to $\overline{c}_{ij}^{pq}$, $\overline{a}_i^{pq}$ and $\overline{b}_j^{pq}$.

1) Solution procedure for classical transportation problem $(t=1)$: $\overline{x}_{ij}^{11} = [x_{ij}^1, x_{ij}^1] = x_{ij}^1$.
2) Solution procedure for transportation problem with Interval numbers $(t=2)$: $\overline{x}_{ij}^{12} = [x_{ij}^1, x_{ij}^2]$ [11].
3) Solution procedure for transportation problem with triangular fuzzy numbers $(t=3)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3) = (x_{ij}^2, \overline{x}_{ij}^{13})$ [11].
4) Solution procedure for transportation problem with trapezoidal fuzzy numbers $(t=4)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4) = (\overline{x}_{ij}^{23}, \overline{x}_{ij}^{14})$ [11].
5) Solution procedure for transportation problem with pentagonal fuzzy numbers $(t=5)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5) = (x_{ij}^3, \overline{x}_{ij}^{24}, \overline{x}_{ij}^{15})$.
6) Solution procedure for transportation problem with hexagonal fuzzy numbers $(t=6)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6) = (\overline{x}_{ij}^{34}, \overline{x}_{ij}^{25}, \overline{x}_{ij}^{16})$.
7) Solution procedure for transportation problem with heptagonal fuzzy numbers $(t=7)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7) = (x_{ij}^4, \overline{x}_{ij}^{35}, \overline{x}_{ij}^{26}, \overline{x}_{ij}^{17})$.
8) Solution procedure for transportation problem with octagonal fuzzy numbers $(t=8)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7, x_{ij}^8) = (\overline{x}_{ij}^{45}, \overline{x}_{ij}^{36}, \overline{x}_{ij}^{27}, \overline{x}_{ij}^{18})$.
9) Solution procedure for transportation problem with nonagonal fuzzy numbers $(t=9)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7, x_{ij}^8, x_{ij}^9) = (x_{ij}^5, \overline{x}_{ij}^{46}, \overline{x}_{ij}^{37}, \overline{x}_{ij}^{28}, \overline{x}_{ij}^{19})$.
10) Solution procedure for transportation problem with decagonal fuzzy numbers $(t=10)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7, x_{ij}^8, x_{ij}^9, x_{ij}^{10}) = (\overline{x}_{ij}^{56}, \overline{x}_{ij}^{47}, \overline{x}_{ij}^{38}, \overline{x}_{ij}^{29}, \overline{x}_{ij}^{110})$.
11) Solution procedure for transportation problem with hendecagonal fuzzy numbers $(t=11)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7, x_{ij}^8, x_{ij}^9, x_{ij}^{10}, x_{ij}^{11}) = (x_{ij}^6, \overline{x}_{ij}^{57}, \overline{x}_{ij}^{48}, \overline{x}_{ij}^{39}, \overline{x}_{ij}^{210}, \overline{x}_{ij}^{111})$.
12) Solution procedure for transportation problem with dodecagonal fuzzy numbers $(t=12)$: $\tilde{x}_{ij} = (x_{ij}^1, x_{ij}^2, x_{ij}^3, x_{ij}^4, x_{ij}^5, x_{ij}^6, x_{ij}^7, x_{ij}^8, x_{ij}^9, x_{ij}^{10}, x_{ij}^{11}, x_{ij}^{12}) = (\overline{x}_{ij}^{67}, \overline{x}_{ij}^{58}, \overline{x}_{ij}^{49}, \overline{x}_{ij}^{310}, \overline{x}_{ij}^{211}, \overline{x}_{ij}^{112})$.

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Conflicts of Interest

All authors have contributed equally in this work. The authors declare that there is no conflict of interest for this publication.

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