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An Innovative Inverse Model of Network Data Envelopment Analysis

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Abstract

Sensitivity analysis in optimization problems is important for managers and decision maker to introduce different strategies. Data Envelopment Analysis (DEA) is a method based on mathematical programming to evaluate the efficiency of a set of Decision-Making Units (DMUs). Due to the importance of sensitivity analysis in an optimization problem, a development of DEA model called inverse model in DEA is presented. The purpose of this model is to analyze the sensitivity of some inputs or outputs to changes in some other inputs or outputs of the unit under evaluation, provided that the amount of efficiency remains constant or improves at the discretion of the manager. In this research, for the first time, we introduce the inverse model in DEA with network structure. In fact, we examine the extent to which the input parameters are likely to change based on the presuppositions of the problem, for the output changes that are applied as the manager desires. One of the key points of this research is that to make the modeling more consistent with reality, the leader-follower method was used in estimating the parameters in the network. In addition, the opinions of the system manager and the decision maker, who have full control over the system under their management, are included in this modeling to estimate the desired values. Another feature of this modeling is the consideration of uncontrollable factors in the inverse model in DEA with network structure. Finally, using a numerical example, the results obtained are analyzed based on the proposed model.

Keywords: Network data envelopment analysis, Inverse data envelopment analysis model, Multi objective.

1 | Introduction

One of the famous techniques for the efficiency evaluation of a set of Decision-Making Units (DMUs) with multiple inputs and outputs is Data Envelopment Analysis (DEA). DEA was first introduced by Charnes et al. [1] based on Farrell efficiency measure. The model proposed by Charnes et al. [1], called CCR, was based on the assumption of constant returns to scale. Later, the BCC model was introduced by Banker et al. [2] assuming a variable return to scale. This technique has attracted the attention of many researchers both in theory and practice [3]. Also, other researchers including Emrouznejad et al. [4] and Kaffash et al. [5] reviewed the research conducted in the field of DEA. In traditional DEA models in which the DMU is considered irrespective of the interaction between its processes, a unit may be efficient while its components and processes are inefficient. Because of the weakness of traditional models in considering the internal structure of units, great efforts have been

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made by the researchers to develop the traditional models so that they can examine the internal structure of multiplier units [6]. To troubleshoot the traditional and independent models, Fare and Grosskopf [7] introduced network DEA models that examine the operation of component processes in estimating the efficiency of the system under evaluation. In numerous studies, the researchers have examined the factors affecting the value of efficiency. Their purpose was to study and estimate the effect of factors affecting the amount of efficiency, with the assumption that the efficiency of the unit under evaluation does not change. These studies are known as inverse DEA. Wei et al. [8] introduced inverse DEA models and presented the first model in this field. They posed the following questions and sought to answer them: 1) if all or some of the outputs of a DMU increase, how much will the inputs of this unit increase (over-consumption of input) while the efficiency of the DMU remains the same?, and 2) if all or some of the inputs of a DMU increase, how much will the outputs of this unit increase (overproduction) without any changes in the efficiency of the DMU? They identified the necessary and sufficient conditions in order to keep the efficiency of the unit under evaluation constant. Following this strand of research, Yan et al. [9] adopted a different approach to examining the models of inverse DEA and answered the questions posed by Wei et al. [8]. To answer the questions posed by Wei et al. [8], Jahanshahloo et al. [10] considered a certain percentage of efficiency improvement for the DMUs and proposed several models accordingly. Also, Hadi Vancheh et al. [11] introduced the necessary and sufficient conditions to answer the questions posed by Wei et al. [8], adopting a different approach. To examine inverse DEA models, Lertworasirikul et al. [12] investigated the extent to which did the outputs (inputs) of this DMU change with no changes in the efficiency when all or some of the inputs (outputs) of a DMU increased? According to DEA literature, in DEA models with network structure in various practical examples, there exist certain uncontrollable factors. As a result, considering these factors in inverse DEA models in network structure can also be very important.

Jahanshahloo et al. [13] used Russell's non-radial model to estimate inputs or outputs in inverse DEA. Mirsalehi et al. [14] proposed a measurement model based on directional slack variables in inverse DEA in a new production possibility set. Ghiyasi [15] modified the model introduced by Lertorasirikul et al. [12] and prposed a new model. Hadi Venche et al. [16], based on the models they introduced in inverse DEA, presented a new model with a different approach to examining inverse DEA models. Ghiyasi [17] introduced inverse DEA models based on profit and cost efficiency. Ghobadi [18] used advanced Russell measurement in the presence of fuzzy data in inverse DEA. Amin et al. [19] used a multi-objective problem combination for inverse DEA. They introduced a method for target setting by considering the models of inverse DEA and ideal programming. Wegner et al. [20] tried to minimize greenhouse gas emissions using inverse DEA models and provided a practical example in this field. Ghiyasi and Khoshfetrat [21] maintained the constant relative efficiency values by providing inverse DEA with inaccurate data. In addition, Ghiyasi and Zhou [22] proposed an inverse semi-oriented radial DEA approach to evaluating efficiency with negative data. Gatimbu et al. [3] evaluated the environmental efficiency of the tea industry in Kenya using the inverse DEA approach. Gerami et al. [23] proposed an inverse DEA model for the purpose of restructuring the company based on value efficiency.

In the present study, a new inverse network DEA model is presented. In this approach, by considering the changes in the outputs determined by the manager, assuming a constant efficiency value, the purpose is to estimate the amount of inputs or intermediate outputs in a network, which is a new and practical topic. Therefore, the goal is also to consider these types of indicators in network models in inverse DEA.

The rest of the paper is organized as follows: In the next section, the prerequisites of presenting the main model will be provided. The main ideas and modeling proposed in this study are presented in section three. In section four, the features and the benefits of the proposed model will be explained by giving numerical examples. Finally, section five includes conclusions and suggestions.

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2 | Inverse Model of Data Envelopment Analysis with Network Structure

In recent years, DEA has been one of the key tools in the area of performance analysis of organizations and activities in various fields. This has resulted in the expansion of the applications of this technique with increasing growth in its theory along with the introduction of more advanced models. One such model is the inverse model in DEA which was first proposed by Wei et al. [8]. They introduced this model for the estimation of input or output after making changes to the outputs or inputs while the amount of efficiency remains constant. After being introduced in the literature, this model quickly drew the attention of researchers and has then been modified and developed from different aspects [19].

The main purpose of this section is to introduce an inverse model of DEA with network structure. To do so, we first consider a two-stage series network in *Fig. 1*. It should be mentioned that since this study is the first attempt in DEA literature, we decided to consider this network. In future studies, different types of network models can be considered for evaluation using inverse DEA models.

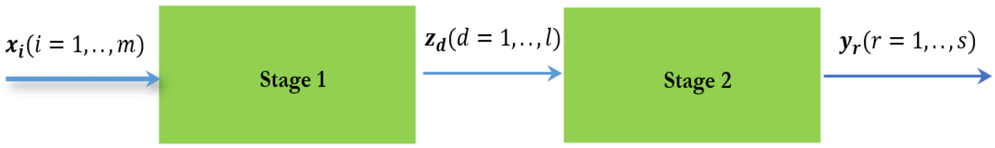

Fig. 1. Series network model with two stages and intermediate products.

Let's assume $DMU_j(j = 1, \dots, n)$ as the jth DMU, which includes inputs, intermediate products, and the outputs. Also, assume that the input, output, and intermediate vectors of DMU_j are as follows:

$$\begin{aligned} x_j &= (x_{1j}, \dots, x_{mj}), \\ z_j &= (z_{1j}, \dots, z_{fj}), \\ y_j &= (y_{1j}, \dots, y_{sj}). \end{aligned} \quad (1)$$

To evaluate the technical efficiency of the network introduced in *Fig. 1*, we consider the input-oriented radial model with constant returns to scale. Therefore, according to the changes desired by the manager or the decision maker, which is to increase the outputs, given a constant or increasing efficiency at the discretion of the manager, the purpose is to estimate input values in the network.

Further assume that DMU_o in which $o \in \{1, \dots, n\}$ is the unit under evaluation. λ_j^1 and λ_j^2 for each j represents the corresponding intensity coefficients of the first and the second components of the network, respectively. Also, θ is a free-sign variable, the function of which is to produce a radial reduction in the independent input of network x. Consider the DEA model with network structure in the input-oriented as follows:

$$\begin{aligned} &\min\ \theta \\ &\text{s.t.}\quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le \theta x_{io}, && i = 1, \dots, m, \\ &\textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge z_{do}, && d = 1, \dots, l, \\ &\textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le z_{do}, && d = 1, \dots, l, \\ &\textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}, && r = 1, \dots, s, \\ &\lambda_j^1 \ge 0,\ \lambda_j^2 \ge 0, && j = 1, \dots, n. \end{aligned} \quad (2)$$

After solving *Model (2)*, it is concluded that θ^* represents the amount of technical efficiency that indicates the maximum radial reduction of the input until it reaches the efficiency limit. Now, if the manager or the decision maker intends to increase the outputs from y to y^N, to estimate the independent network input, i.e., x, *Model (3)* is introduced. Note that in this model θ^* is the efficiency obtained from *Model (2)*. The main purpose of *Model (3)* is to estimate all the components of the independent input vector in the network, namely, x.

But because minimizing each of the components x_i $(i = 1, \ldots, m)$ yields a model with m objective functions, *Model (3)* is a multi-objective optimization model. In *Model (3)*, the corresponding variable vector of the input vector x_i $(i = 1, \ldots, m)$ is vector β which has m components as $\beta = (\beta_1, \ldots, \beta_m)$. According to Wei et al. [8], to estimate inputs, for each m input component, the condition $\beta_i \geq x_{io}$ is considered for each i.

$$
\begin{aligned}
\min \quad & (\beta_1, \ldots, \beta_m) \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^* \beta_i, \quad i = 1, \ldots, m, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, \quad d = 1, \ldots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, \quad d = 1, \ldots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, \quad r = 1, \ldots, s, \\
& \beta_i \geq x_{io} \geq 0, \quad i = 1, \ldots, m, \\
& \lambda_j^1 \geq 0\,, \ \lambda_j^2 \geq 0, \quad j = 1, \ldots, n, k = 1,2.
\end{aligned} \tag{3}
$$

We convert the multi-objective *Model (3)* to the single-objective *Model (4)* in which all of the components of vector x are minimized.

$$
\begin{aligned}
\min \quad & \textstyle\sum_{j=1}^{m} \beta_i \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^* \beta_i, \quad i = 1, \ldots, m, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq z_{do}, \quad d = 1, \ldots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq z_{do}, \quad d = 1, \ldots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq y_{ro}^N, \quad r = 1, \ldots, s, \\
& \beta_i \geq x_{io} \geq 0, \quad i = 1, \ldots, m, \\
& \lambda_j^1 \geq 0\,, \ \lambda_j^2 \geq 0, \quad j = 1, \ldots, n, k = 1,2.
\end{aligned} \tag{4}
$$

If the manager is interested in conducting the evaluation in such a way that he can know whether he could have more efficiency when there was more production with minimal increase in inputs, he can select a value from the range of the defined values to improve efficiency and estimate the inputs under these conditions. The efficiency improvement range is:

$$0 \leq \theta \leq 1 - \theta^*, 0 < \theta^* \leq 1. \tag{5}$$

Therefore, if one intends to keep efficiency constant, he or she just needs to consider any increase (improvement) in the efficiency value to be zero, i.e., $\theta = 0$. If the efficiency of the unit under evaluation is 1, i.e., $\theta^* = 1$, then it is concluded from the above relation that $\theta = 0$. Otherwise, if $\theta^* < 1$, then the manager can maximize the efficiency value up to 1- θ^*, i.e., the new efficiency value is assumed to be $\theta^* + \theta$, where $\theta^* + \theta = 1$. So, we will have *Model (6)*.

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{m} \beta_i \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le (\theta^* + \theta)\beta_i, \quad i = 1, \dots, m, \quad && \text{(a)} \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge z_{do}, \quad d = 1, \dots, l, && \text{b)} \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le z_{do}, \quad d = 1, \dots, l, && \text{c)} \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}^N, \quad r = 1, \dots, s, && \text{d)} \\
& \beta_i \ge x_{io} \ge 0, \quad i = 1, \dots, m, && \text{e)} \\
& \lambda_j^1 \ge 0, \ \lambda_j^2 \ge 0, \quad j = 1, \dots, n. && \text{(f)}
\end{aligned} \tag{6}
$$

Note that in *Model (6)*, the intermediate products are not assumed to be variable. In this case, a new value may not be derived to estimate x because if y increases to y^N then λ_j^2 can take different values depending on the new output value. But since λ_j^2 should also help the *Constraint (C)* to hold, λ_j^2 cannot take a different value. In this case, no estimate can be obtained for x.

Therefore, to introduce a more comprehensive model, we present *Model (7)*. In *Model (7)*, the intermediate products are also considered to be variable. After increasing y to y^N by the decision maker, in *Model (7)*, the goal is to estimate the values corresponding to z and x provided that the efficiency value remains constant or improves. Thus, the corresponding vector of the input variable $x = (x_1, \dots, x_m)$ is the vector $\beta = (\beta_1, \dots, \beta_m)$ and the corresponding variable vector of the intermediate product $z = (z_1, \dots, z_l)$ is the vector $\gamma = (\gamma_1, \dots, \gamma_l)$.

Borrowing from Wei et al. [21], to estimate inputs and intermediate products, the conditions $\beta_1 \ge x_{io}$ and $\gamma_d \ge z_{do}$ are assumed to hold for each input component i=1,…,m and each intermediate product component $d = 1, \dots, l$. Note that minimizing each of the components $x_i (i = 1, \dots, m)$ and $z_d (d = 1, \dots, l)$ results in a model with m+l objective functions.

$$
\begin{aligned}
& \min(\gamma_1, \dots, \gamma_l) \\
& \min(\beta_1, \dots, \beta_m) \\
\text{s.t.} \quad & \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le (\theta^* + \theta)\beta_i, \quad i = 1, \dots, m, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge \gamma_d, \quad d = 1, \dots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le \gamma_d, \quad d = 1, \dots, l, \\
& \textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}^N, \quad r = 1, \dots, s, \\
& \beta_i \ge x_{io} \ge 0, \quad i = 1, \dots, m, \\
& \gamma_d \ge z_{do} \ge 0, \quad d = 1, \dots, l, \\
& \lambda_j^1 \ge 0, \lambda_j^2 \ge 0, \quad j = 1, \dots, n.
\end{aligned} \tag{7}
$$

Thus, the single-objective model corresponding to *Model (7)* is *Model (8)*, in which all of the components of the vectors x and z are minimized. Consider *Model (8)*:

$$
\begin{aligned}
&\min \ \textstyle\sum_{j=1}^{m} \beta_i + \sum_{d=1}^{l} \gamma_d \\
&\text{s.t.} \quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le (\theta^* + \theta)\beta_i, \qquad i = 1, \ldots, m, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge \gamma_d, \qquad d = 1, \ldots, l, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le \gamma_d, \qquad d = 1, \ldots, l, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}^N, \qquad r = 1, \ldots, s, \\
&\beta_i \ge x_{io} \ge 0, \qquad i = 1, \ldots, m, \\
&\gamma_d \ge z_{do} \ge 0, \qquad d = 1, \ldots, l, \\
&\lambda_j^1 \ge 0, \ \lambda_j^2 \ge 0, \qquad j = 1, \ldots, n.
\end{aligned}
\tag{8}
$$

It should be mentioned that always in real examples and case studies, it is possible for the changes to be both increasing and decreasing. Also, during the estimation of the amounts of inputs, increasing the inputs may not always be intended. So, we introduce *Model (7)* more generally as follows. This model can be used to analyze the sensitivity of the obtained optimal solution.

$$
\begin{aligned}
&\min \sum_{j=1}^{m} \beta_i + \sum_{d=1}^{l} \gamma_d \\
&\text{s.t.} \quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \le (\theta^* + \theta)(\beta_i + x_{io}), \qquad i = 1, \ldots, m, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \ge \left(z_{dj} + \gamma_d\right), \qquad d = 1, \ldots, l, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \le \left(z_{dj} + \gamma_d\right), \qquad d = 1, \ldots, l, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \ge y_{ro}^N, \qquad r = 1, \ldots, s, \\
&x_{io} + \beta_i \ge 0, \qquad i = 1, \ldots, m, \\
&z_{do} + \gamma_d \ge 0, \qquad d = 1, \ldots, l, \\
&\lambda_j^1 \ge 0, \ \lambda_j^2 \ge 0, \qquad j = 1, \ldots, n, \\
&\beta_i, \gamma_d \ \text{free in sign}, \qquad i = 1, \ldots, m, d = 1, \ldots, l.
\end{aligned}
\tag{9}
$$

As can be seen in *Model (8)*, the estimation of the input vectors and intermediate products was intended to increase. That is, the estimates of the values of these vectors were more than the initial values. However, in *Model (9)*, changes are generally considered both for the input vectors and intermediate products. Thus, the estimated values for these vectors can be less than the initial values. In addition, in order for the data distribution in the estimated vector not to go beyond the manager's expectations and thus not to lead to unrealistic and non-operational results in the practical examples, inverse DEA model with network structure adopting the leader and follower approach is considered. In this approach, the purpose is to estimate the vectors in a specific interval that has already been provided by the decision maker.

Model (9) for the two-stage process is shown in *Fig. 1*. If we wish to consider the above process for each stage separately, we should proceed as follows:

First, the following model is solved:

$$\begin{aligned}
&\min \textstyle\sum_{d=1}^{l} \gamma_d \\
&\text{s.t.} \quad \textstyle\sum_{j=1}^{n} \lambda_j^2 z_{dj} \leq \; z_{do} + \gamma_d), \quad d = 1, \ldots, l, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^2 y_{rj} \geq \varphi^* y_{ro}^N, \quad r = 1, \ldots, s, \\
&z_{do} + \gamma_d \geq 0, \quad d = 1, \ldots, l, \\
&\lambda_j^2 \geq 0, \quad j = 1, \ldots n, \\
&\gamma_d \text{ free in sign}, \quad d = 1, \ldots, l.
\end{aligned} \tag{10}$$

Due to the increasing output applied by the decision maker, the estimation of each component of the intermediate production vector is obtained as $(z_{do} + \gamma_d^*), d = 1, \ldots, l$. Now, the intermediate products change from z to $z^N = z + \gamma^*$. As a result, *Model (11)* is defined as follows to estimate the independent inputs of the network's first stage provided that the input-oriented technical efficiency of the first component of the network is constant or improved.

$$\begin{aligned}
&\min \; \textstyle\sum_{j=1}^{m} \beta_i \\
&\text{s.t.} \quad \textstyle\sum_{j=1}^{n} \lambda_j^1 x_{ij} \leq \theta^* \; x_{io} + \beta_i), \quad i = 1, \ldots, m, \\
&\textstyle\sum_{j=1}^{n} \lambda_j^1 z_{dj} \geq \; z_{do} + \gamma_d^*), \quad d = 1, \ldots, l, \\
&x_{io} + \beta_i \geq 0, \quad i = 1, \ldots, m, \\
&\lambda_j^1 \geq 0, \quad j = 1, \ldots, n, \\
&\beta_i \text{ free in sign}, \quad i = 1, \ldots, m.
\end{aligned} \tag{11}$$

From the optimal solution obtained from the *Model (11)*, the estimated input vector with components $x_{io} + \beta_i^*, i = 1, \ldots, m$ is obtained. Therefore, under these conditions, the changes of input and intermediate indices can be calculated independently for each stage according to the changes of the final output index.

3 | Application

The purpose of this section is to show the importance of the models presented in the previous section. To this aim, we first use a numerical example to demonstrate the proposed approach. To do so, the models presented above are used to evaluate the efficiency of hospitals in Iran as a practical example. In this application, 10 health care systems with two inputs, two outputs, and one intermediate product corresponding to *Fig. 1* are taken into consideration.

Input variables are: (I_1) personnel costs which show salaries and wages of employees working in each HCS; (I_2) administrative costs which show the costs of purchasing medical equipment in the hospital to serve the patients. Intermediate variable is (Z), the cost of doctors' benefits and case-based payments (including rewards and fee-for-service allocated to physicians to provide the best possible service). Output variables are: (O_1) inpatients' medical bill total which shows the amount of money received to serve the hospitalized patients in each HCS; (O_2) outpatients' medical bill total which shows the amount of money received to serve the patients treated on an outpatient basis. The information of these indexes is given in *Table 1*.

DMU	I_1	I_2	O_1	O_2	Z
1	15	98	45	12	45
2	24	12	44	32	65
3	31	11	75	65	34

Table 1. Data set.

4	20	34	37	15	62
5	18	23	86	61	87
6	16	16	24	23	12
7	14	76	24	54	62
8	75	20	98	18	49
9	25	17	73	24	82
10	72	73	24	61	61

After the output y y is modified into y^N by the decision maker, the enhanced outputs are given in *Table 2*.

Table 2. New outputs.

DMU	O_1^N	O_2^N	DMU	O_1^N	O_2^N
1	46	14	6	22	24
2	45	33	7	25	55
3	73	63	8	95	16
4	38	16	9	74	23
5	85	60	10	26	60

The values of the inputs are estimated according to *Models (2)* and *(3)*. It is evident from the results of *Table 3* that the values of the inputs were not estimated under the assumption that the intermediate products were variable. The estimated inputs derived from *Model (3)* are the initial inputs.

Table 3. Efficiency and estimated inputs.

DMU	Efficiency	I_1^*	I_2^*
1	0.62	15	98
2	1	24	12
3	0.57	31	11
4	0.64	20	34
5	1	18	23
6	0.18	16	16
7	0.92	14	76
8	0.45	75	20
9	1	25	17
10	0.2	72	73

In *Model (4)*, the intermediate products are also considered to be variable and the purpose is to estimate the values of z and x assuming a constant efficiency. The results are shown in *Table 4*.

DMU	Efficiency	I_1^*	I_2^*	$Z_{\square}^*$
1	0.62	15	98	45
2	1	24	12	65

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Table 4. Estimated inputs and intermediate products from Model (4).

3	0.57	31	11	34
4	0.64	20	34	62
5	1	18	23	87
6	0.18	16	16	12
7	0.92	14	76	62
8	0.45	75	20	49
9	1	25	17	82
10	0.2	72	73	61

The results of estimating the inputs and the intermediate products in the example mentioned above are given in *Table 5* by solving *Model (5)* in which the estimation of the vectors are not only intended to increase the values.

Table 5. Estimated input and intermediate products from Model (5).

DMU	Efficiency	I_1^*	I_2^*	$Z_{\square}^*$	Efficiency From Estimated Data
1	0.62	6.95	8.88	20.85	0.62
2	1	4.22	5.39	20.4	1
3	0.57	12	15.33	33.09	0.57
4	0.64	5.56	7.1	17.23	0.64
5	1	7.97	10.19	38.53	1
6	0.18	14.2	18.15	12.52	0.18
7	0.92	6.48	8.28	28.7	0.92
8	0.45	19.7	25.17	43.07	0.45
9	1	6.94	8.87	33.55	1
10	0.2	31.71	40.51	31.3	0.2

The results obtained from solving *Models (6)* to *(9)* are summarized in *Table 6*. *Table 6* shows the values of the output-oriented efficiency of the second component of the network and the input-oriented efficiency of the first component in *Models (6)* and *(7)*. Also, using *Model (8)*, the estimation of the intermediate products when the outputs increase by the manager is provided.

Model (9) estimates the inputs by considering the estimated intermediate products as the modified output for the first component of the network. In *Models (8)* and *(9)* for the estimation of the intermediate products and inputs, boundaries are prescribed by the manager to control the distribution of the estimated values.

Table 6. The results of estimating inputs and intermediate products using the method with leader and follower structure.

DMU	Effi-Out	$Z_{\square}^*$	Effi-Inp	I_1^*	I_2^*
1	2.21	46.00	0.63	15.00	88.20
2	3.26	66.48	1.0	21.60	12.85
3	1.00	33.09	0.56	27.90	11.00
4	3.70	63.68	0.66	20.00	30.60
5	2.23	85.99	0.99	18.00	23.00
6	1.00	12.52	0.19	14.40	16.93
7	2.20	63.15	0.93	14.00	68.40
8	1.10	47.50	0.44	67.50	20.00
9	2.48	83.12	1.0	22.50	18.46
10	1.92	60.00	0.20	64.80	77.19

In the third column, the middle index and in the fifth and the sixth columns, the input indices are estimated using *Model (9)*. It is important to note that according to the fourth column of *Table 6*, the efficiency of DMUs remains constant after the implementation of the model. In *Model (9)* because the variables β_i, γ_d

are free sign variables, the estimated points for the input and intermediate indices given in *Table 6* are higher than some of their initial values for some DMUs and less than their initial values for some others. For example, the estimated value for the middle index of the first DMU is 46 and its initial value is 45. The intermediate index estimated for the third DMU is 33.09 and the initial value is 34. This model seeks to find the minimum changes in the input and intermediate indicators so that after the changes applied to the outputs by the management, the efficiency remains constant, which is confirmed according to the results obtained from the implementation of the model.

4 | Conclusion

In this paper, models in inverse DEA in network structure were presented. By taking into account the changes in the outputs or inputs specified by the manager, assuming a constant efficiency value, the amount of inputs, outputs, or intermediate products in a network was estimated. Finally, the presented models were implemented on a numerical example and the results were analyzed. It can also be important to consider the same issue for estimating inputs, outputs, or intermediates by considering the profitability challenge. In this study, an inverse two-stage network DEA model was presented that can be extended to three-stage or more networks. Either we can extend the proposed network model to fuzzy or negative data, or we can change the structure of the network model, for example using a parallel or relational structure. In addition, in this study, we used such indicators as undesirable output, each of which can be considered as a future research.

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Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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