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Cost Efficiency in the Presence of Time Dependent Prices

Jafar Pourmahmoud[1],[*] iD **, Naser Kaheh[1], Farhad Hosseinzadeh Lotfi[2]**

[1] Department of Applied Mathematic, Azarbaijan Shahid Madani University, Tabriz, Iran; pourmahmoud@azaruniv.ac.ir; n.kaheh@azaruniv.ac.ir.

[2] Department of Mathematical, Science and Research Branch, Islamic Azad University, Tehran, Iran; farhad@hosseinzadeh.ir.

Abstract

In traditional cost-efficiency models, inputs and outputs, as well as input prices were known as constant values for each decision-making unit In our daily applications, however, market entry prices vary at different times. In other words, input prices for Decision-Making Units (DMUs) are time dependent. Traditional methods cannot calculate the cost efficiency of DMUs with time-dependent prices. This paper proposes a new method to calculate the cost efficiency of DMUs in the presence of time-dependent prices. The proposed model is a parametric programming problem model depending on time. In the presented model, the inputs and outputs are functions in terms of time, which is not present in the models introduced by other researchers. New definitions for time-dependent cost efficiency have also been introduced. The cost efficiency of DMUs is measured over a given time and the units are ranked according to the time obtained. Finally, a numerical example has been presented to illustrate the proposed method.

Keywords: Data envelopment analysis, Cost efficiency, Time Dependent prices, Ranking.

1 | Introduction

Data Envelopment Analysis (DEA), first introduced by Charnes et al. [4], is a non-parametric method for evaluating the performance of Decision-Making Units (DMUs). With the help of DEA, the efficiency of the DMUs can be calculated and the DMUs can be ranked. Many studies have been conducted in this area, including Banker et al. [1], Edalatpanah [7], Khodabakhshi and Cheraghali [17], and Maghbouli and Moradi [20] Managers, in addition to evaluating DMUs, are always looking for all ways of manufacturing their products with minimal costs. In practice, DMUs can be valued in terms of costs, profits, or revenues if input and output prices are available. In fact, in cost efficiency models, the ability to yield current outputs is evaluated at minimum cost.

The concept of cost efficiency was first introduced by Farrell [11] and then by Färe et al. [9] used the linear programming model to develop cost efficiency. They defined the cost-effectiveness of a DMU as the ratio between the minimum production cost and the actual observed cost. Tone [29] improved

the Färe et al. [9] by evaluating DMUs using cost-based production possibility rates instead of traditional production possibility rates. Other studies in this area include Kholmuminov and Wright [18], Haralayya and Aithal [12], [13].

Traditional methods (such as Färe et al. [9] and Tone [29]) of the cost efficiency of different DMUs have been calculated when all data, including inputs, outputs and prices of inputs, are determined precisely. However, in practice, the values for the outputs or inputs or their prices may be uncertain.

Recently, various uncertain data have been introduced, including fuzzy data, interval data, and stochastic data. In addition, many studies have been conducted to calculate the efficiency of DMUs, as well as the cost efficiency, profit efficiency, and revenue efficiency of DMUs in the presence of uncertain data. Some of these studies are discussed here.

In recent years, many researchers have studied the use of fuzzy theory for cost efficiency. Jahanshahloo et al. [16] examined for the first time the assessment of cost efficiency considering the fuzzy DEA. Several studies have subsequently been carried out in this area, including Puri and Yadav [25] Pourmahmoud and Sharak [23], [24]. Camanho and Dyson [3], and Hosseinzadeh Lotfi et al. [15] are the first ones to calculate DMUs cost efficiency in the presence of interval data. Camanho and Dyson [3] calculated the cost efficiency of DMUs while the inputs and outputs values at each DMU were certain and input prices were interval. They calculated the cost efficiency of DMU in which the input values and output values of each DMU as well as the input prices were certain. Sun et al. [27] and Dyvak et al. [6] are the most recent studies in the field of interval data.

There are many models and methods associated with time in DEA, but most of them attempt to study DMU performance over different time periods. These methods can be divided into three categories including productivity index, window analysis and dynamic systems. In the network structure, the Malmquist Productivity Index (MPI), first introduced by the Swedish economist Malmquist [21], is used to measure changes in productivity over time. Dynamic systems are repetitions of the single-period systems which are connected by carryovers, where a single-period system can have any particular structure. Färe and Grosskopf [10] studied this topic earlier. Li et al. [19] studied dynamic prediction of financial distress using Malmquist DEA. Pourmahmoud [22] introduced a new model for ranking DMU based on Dynamic DEA. Data window analysis method was first introduced by Charnes and Cooper [5], as a window analysis. The window analysis method evaluates the performance of each DMU as if it had a different identity at any point in time. In this method, each window consists of a specific number of studies years, beginning with the base year and continuing for the duration of the window. The efficiency values of each DMU are calculated each year, taking into account that the average of the efficiency calculated in this window is the efficiency value of that DMU in this window. By moving the window to a new period (deleting the base year and adding a year at the end of the window), the efficiency values in the new window are calculated for DMUs. Finally, the performance of each DMU is evaluated by comparing each window's efficiency scores to other DMUs over the period.

Another time-related problem is calculating the efficiency of DMUs with time-varying data. Another type of data, called time-dependent data, was first reported by Taeb et al. [28]. They calculated the efficiency of DMUs where the input and output values were time-dependent. In fact, the price of gold, oil, stocks, etc. depends on time. Sometimes the price fluctuations are strong, so the change in cost efficiency over a period of time is very significant. A rapid calculation is required to have a real and updated value for cost efficiency. In this study is to calculate the cost efficiency of DMUs in which market prices for inputs are time dependent and the values of inputs and outputs are certain. Recently, studies have been conducted on cost efficiency, including Fallahnejad et al. [8], Soleimani-Chamkhorami and Ghobadi [26], and Hatami-Marbini and Arabmaldar [14], which none of them is time-dependent data. In this study, we consider inputs and outputs as a function of time. This feature is not present in any of the previous models. This is the superiority of the model presented in this study compared to previous models. Literature gap is given in the table below.

Researches	Method Used
Previous Researches [3], [15], [16], [28]	Calculating the efficiency of DMUs or their cost efficiency in the presence of inputs and outputs time-dependent or fuzzy or interval.
Current study	Calculation of DMUs cost efficiency in the presence of time-dependent prices.
Comparison	In none of the previous models, cost efficiency has been calculated in the presence of time-dependent input prices. In this study, for the first time, input prices are considered as a function of time.

This paper is organized as follows. Section 2 includes an introduction to the traditional cost efficiency from the viewpoint of Färe et al. [9]. Section 3 presents the proposed time-dependent cost efficiency model. In Section 4, a numerical example illustrating the proposed approach is provided. Finally, conclusion is given in Section 5.

2 | Introduction to the Old-Style Cost Efficiency

Suppose DMU_k produces the output $y_k = (y_{1k}, y_{2k}, ..., y_{sk})$ using the input $x_k = (x_{1k}, x_{2k}, ..., x_{mk})$ while all input and output components are nonnegative and $x_k \neq 0$, $y_k \neq 0$. Suppose also that the input price vector for DMU_k is $w_k = (w_{1k}, w_{2k}, ..., w_{mk})$. Therefore, the observed cost by DMU_k is:

$$C_k = \sum_{i=1}^{m} w_{ik} x_{ik}.$$

DMU_k is cost efficient when the observed cost by DMU_k is the lowest cost that can produce y_k. Suppose there are a set of n observations on the DMUs each of which produces s number of outputs by using m number of inputs. Banker et al. [1] defined the production feasibility set for these data in Variable Return to Scale (VRS) mode as follows:

$$T = \{(x,y) : x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, 2, ..., n\}.$$

Färe et al. [9] introduced cost minimization model as follows:

$$C_k^* = \min w_k x$$
$$\text{s.t. } (x, y_k) \in T. \tag{1}$$

Model (1) in VRS mode is written as follows:

$$C_k^* = \min \sum_{i=1}^{m} w_{ik} x_i$$
$$\text{s.t. } \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, i = 1, 2, ..., m,$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, r = 1, 2, ..., s,$$
$$\sum_{j=1}^{n} \lambda_j = 1,$$
$$\lambda_j \geq 0, j = 1, 2, ..., n. \tag{2}$$

If $x^* = (x_1^*, x_2^*, ..., x_m^*)$ is the optimal solution for *Model (2)*, the minimum cost for producing y_k is

$$C_k^* = \sum_{i=1}^{m} w_{ik} x_i^*.$$

Since $(x_k, y_k) \in T$, and C_k is the actually observed cost by DMU_k and C_k^* is also the minimum possible cost for producing the output of DMU_k, so $C_k^* \leq C_k$. DMU_k is cost efficient whenever $C_k^* = C_k$. Färe et al. [9] introduced the cost efficiency of DMU_k as follows:

$$CE_k = \frac{C_k^*}{C_k}.$$

Evidently, $CE_k \leq 1$; and the closer CE_k gets one, the more efficient DMU_k becomes in terms of cost.

3 | Proposed Model for Time Dependent Cost Efficiency

Since in real market ices of gold, oil, and stock etc. often fluctuate over time in some cases, cost efficiency evaluation of DMUs is considered over a period of time. Therefore, the cost efficiency of DMUs that use these inputs would vary by time. Hence, in such a case the estimation would be time dependent. Based on the Färe model, suppose market prices at time t for input i of DMU_k equals $w_{ik}(t)$. $w_{ik}(t)$ is a time dependent function that may be a constant function. Being constant means that the price of the i-th input of DMU_k over a period of time is constant. Therefore the observed cost by DMU_k (at the moment t) is

$$C_k(t) = \sum_{i=1}^{m} w_{ik}(t) x_{ik}. \tag{3}$$

DMU_k is cost efficient at moment t when the observed cost by DMU_k is the lowest cost that can produc y_k.

If market prices of the inputs are time dependent, the minimum cost for producing y_k will also be time dependent. Therefore, *Model (2)* can be rewritten as follows to estimate the cost of DMU_k in VRS mode over t$\in$ [a, b]:

$$
\begin{aligned}
C_k^*(t) = \min &\sum_{i=1}^{m} w_{ik}(t) x_i \\
\text{s.t.} \quad &\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_i, i = 1, 2, \ldots, m, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rk}, r = 1, 2, \ldots, s, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geq 0, j = 1, 2, \ldots, n.
\end{aligned}
\tag{4}
$$

Definition 1. A function of Time Dependent Cost Efficiency (TDCE) for DMU_k, $CE_k(t)$ is defined as follows:

$$CE_k(t) = \frac{C_k^*(t)}{C_k(t)}. \tag{5}$$

Obviously, in *Relation (5)*, $C_k(t) \neq 0$ for all t since the observed cost by a DMU would never be zero. Also with respect to *Model (3)* for all t, $CE_k(t) \leq 1$. If for DMU_k at t_i, $CE_k(t_i) = 1$ then this DMU will be the cost efficient at t_i consequently, DMU_k may be efficient in one moment and inefficient in another moment. Hence, for DMU_k there will be three states. The following definition shows TDCE state of $DMUs$.

Definition 2. In terms of TDCE, the following three states can be considered for DMU_k:

I. If *for all $t \in [a,b]$; $CE_k(t) = 1$,* then DMU_k is named global-efficient based on TDCE.

II. If *there exists $t_1, t_2 \in [a,b]$; $CE_k(t_1) = 1$, $CE_k(t_2) < 1$,* then DMU_k is named local-efficient based on TDCE.

III. If *for all $t \in [a,b]$; $CE_k(t) < 1$,* then DMU_k is named none-efficient based on TDCE.

The following is a graph of the cost efficiency in terms of time for DMUs in all three states:

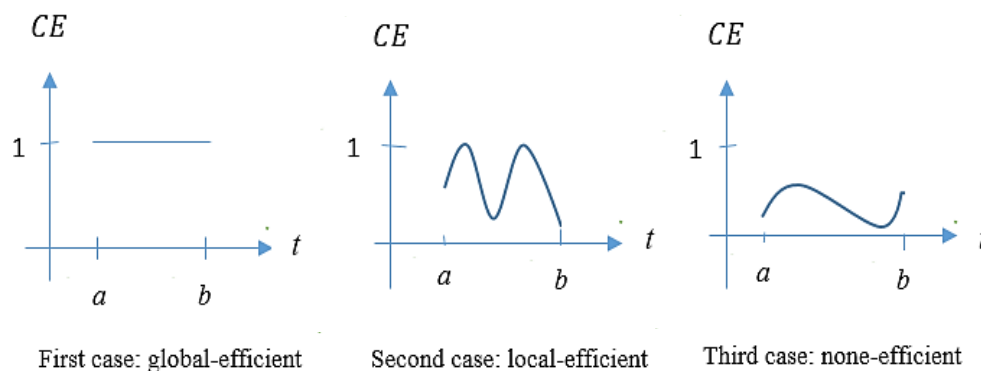First case: global-efficient   Second case: local-efficient   Third case: none-efficient

Fig. 1. Graph of the TDCE for DMUs.

As mentioned above, a more comprehensive definition of TDCE can be introduced and ranking DMUs can also be done using this definition.

Definition 3. The value for the TDCE of DMU_k over $T = [t_1, t_2]$ $(t_1 < t_2)$ which is shown by the symbol CE_k^T can be calculated as follows:

$$CE_k^T = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} CE_k(t)dt. \tag{6}$$

Theorem 1. For every time interval $T = [t_1, t_2]$ $(t_1 < t_2)$ we have $CE_k^T \leq 1$.

Proof: given that for all $t \in T = [t_1, t_2]$, $CE_k(t) \leq 1$, then:

$$\int_{t_1}^{t_2} CE_k(t)dt \leq \int_{t_1}^{t_2} dt \Rightarrow \int_{t_1}^{t_2} CE_k(t)dt \leq t_2 - t_1 \Rightarrow \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} CE_k(t)dt \leq 1 \Rightarrow CE_k^T \leq 1.$$

The geometric proof of the theorem can be stated using the data in *Fig. 2* as follows. In *Definition 3*, the value of $\int_{t_1}^{t_2} CE_k(t)dt$ is the area under the graph of the function $CE_k(t)$ shown by the colored part in *Fig. 2*. $t_2 - t_1$ is also the value of $\int_{t_1}^{t_2} dt$, that is, the area under the graph of the constant function $CE_k(t) = 1$. So the value for CE_k^T is always less than or equal to one.

Proposition 1. In TDCE evaluation of DMUs, at any given moment, there is always a *DMU* whose cost efficiency is one.

Proof: In the calculation of DMUs cost efficiency, in the traditional *Model (2)*, there is always at least a DMU whose cost efficiency equals one. Since time is fixed, when DMUs in the given $t = \alpha$ is evaluated, *Model (4)* becomes the traditional *Model (2)*. So at this moment, there is a DMU whose cost efficiency equals one.
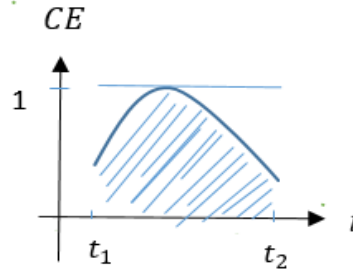
Fig. 2. The cost efficiency versus time.

Corollary 1. In TDCE evaluation of DMUs, there may not be a DMU that is global-efficient, but there is always a DMU that is local-efficient. Therefore, it can be said that in this type of evaluation, at any time interval, for n DMU under evaluation, there is always a DMU that is not totally none-efficient.

Definition 4. In the time interval I = [a, b] for DMU_k, the *Definition 3* is rewritten as follows:

$$CE_k^I = \frac{1}{b-a} \int_a^b CE_k(t)dt. \tag{7}$$

In the specific case $I_0 = [0,1]$, the TDCE value of DMU_k is equal to:

$$CE_k^{I_0} = \int_0^1 CE_k(t)dt. \tag{8}$$

Definition 5. If we have in $T = [t_1, t_2]$ for DMU_k and $DMU_q : CE_q^T < CE_k^T$, then efficiency of DMU_k is higher than efficiency of DMU_q and as a result DMU_k is more efficient than DMU_q. Hence, using this definition, DMUs can be ranked according to their performance.

Solving algorithm for Model (4)

The price of the inputs in the objective function of *Model (4)* is a function of the variable t, which changes continuously. Given the definition of the parametric programming problem, *Model (4)* is also a parametric programming problem. The model can be solved by a parametric problem solving algorithm. The model resolution algorithm in *Model (4)* is proposed by Bazaraa et al. [2]:

1. For t = a, solve the model with the simplex algorithm.
2. Replace the alterations ed by changes in the objective function cost vector using the sensitivity analysis in optimal table extracted from Step 1. In other words, calculate the objective function values and $z_j - c_j$ of the non-basic variables by considering the cost vector and replace them in the row of the optimal table objective function extracted from Step 1. If the final table fails to be unified after the effect of parameter t, unify it.
3. Find the permissible range of parameter t by setting $z_j - c_j \leq 0$ to keep the table optimized. Then increase t until the table loses optimality and select the first available non-basic variable $z_j - c_j > 0$ as the basic input variable.
4. Repeat Step 3 until for every t and for all non-basic variables $z_j - c_j \leq 0$. Then the optimal solution is obtained.

4 | Numerical Examples

Three numerical examples in different modes are presented in this section to illustrate the proposed method. All three examples measure TDCE functions and TDCE values for five two-input, one-output DMUs. The input values for the DMUs have not changed in all three examples. In the first example, the output of all DMUs is one. In the second example, we assume that the input prices are different for each DMU. Finally, in the third example, we not only look at input prices differently, but also at different outputs. The proposed method had good results for the three cases.

Example 1. Suppose that there are five DMUs with two inputs and one output. *Table 2* shows the crispt inputs of DMUs. Additioanly, assume that the output of all DMUs is the same value as $y = 1$ and the vector of time dependent prices for inputs is the same for all DMUs as $(1+5t, t+1)$.

Table 2. The inputs for 5 DMUs.

DMUs / Inputs	A	B	C	D	E
Input 1	5	3	4	2	2
Input 2	1	2	5	4	6

The DMUs in *Table 2* are shown in *Fig. 3* in the input space. *Fig. 3* shows that A, B and D are strongly efficient, E is weakly efficient and C is inefficient.
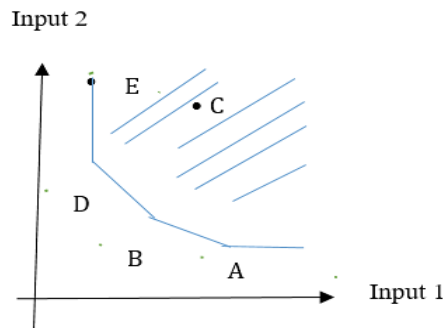

Fig. 3. Show DMUs in inputs space.

Since the output of all DMUs is assumed to be equal to 1, the minimum cost to produce this output must be the same. *Model (4)* is applied for DMUs in *Table 2* and the following results are obtained;

$$C_A^*(t) = C_B^*(t) = C_C^*(t) = C_D^*(t) = C_E^*(t) = \begin{cases} 17t + 5; 0 \le t \le 1/3, \\ 14t + 6; t > 1/3. \end{cases}$$

By considering the inputs of each DMU and their costs and using *Eq. (3)*, the observed cost by each DMUs is obtained as follows:

$$C_A(t) = 26t + 6,$$

$$C_B(t) = 17t + 5,$$

$$C_C(t) = 25t + 9,$$

$$C_D(t) = 14t + 6,$$

$$C_E(t) = 16t + 8.$$

By applying *Eq. (5)* for each DMUs, the TDCE function of each DMU is:

Cost efficiency in the presence of time dependent prices

$$CE_A(t) = \begin{cases} \dfrac{17t+5}{26t+6}, & 0 \le t \le 1/3, \\[4mm] \dfrac{14t+6}{26t+6}, & t > 1/3. \end{cases}$$

$$CE_B(t) = \begin{cases} 1, & 0 \le t \le 1/3, \\[4mm] \dfrac{14t+6}{17t+5}, & t > 1/3. \end{cases}$$

$$CE_C(t) = \begin{cases} \dfrac{17t+5}{25t+9}, & 0 \le t \le 1/3, \\[4mm] \dfrac{14t+6}{25t+9}, & t > 1/3. \end{cases}$$

$$CE_D(t) = \begin{cases} \dfrac{17t+5}{14t+6}, & 0 \le t \le 1/3, \\[4mm] 1, & t > 1/3. \end{cases}$$

$$CE_E(t) = \begin{cases} \dfrac{17t+5}{16t+8}, & 0 \le t \le 1/3, \\[4mm] \dfrac{14t+6}{16t+8}, & t > 1/3. \end{cases}$$

Cost efficiency of DMUs can be easily calculated at any time. As an example, in $t = 0.1$ cost efficiency values with four decimal places for A, B, C, D and E are 0.9139, 1.0000, 0.5547, 0.9054 and 0.5833, respectively.

Suppose that DMUs are evaluated in interval $I_0 = [0,1]$. In this case, the TDCE value of DMUs in I_0 is calculated by applying *Eq. (8)* to *4* decimal places are shown in *Table 3*.

Table 3. The TDCE value of DMUs over I_0 with the same time dependent price.

DMU	A	B	C	D	E
$CE_k^{I_0}$, $k = A,B,C,D,E$	0.6970	0.9625	0.5965	0.9775	0.7890

According to *Table 3*, none of the DMUs are global-efficient and the DMUs were fully ranked. The ranking of these DMUs is as follows: D, B, E, A, C.

Considering TDCE functions of each DMU, D and B are local-efficient and A, C and E are non-efficient. Although it is not necessary to have a DMU whose TDCE value is equal to one in evaluating DMUs in terms of TDCE, this may also happen. If we consider the time interval under evaluation, instead of $I_0 = [0,1]$, the time interval [0, 1/3], B will be global-efficient. According to the results of *Table 3*, the ranking of the DMUs in terms of TDCE may change compared to their traditional ranking. For example, in the traditional model and by considering the same outputs $y = 1$, DMU_A is strong efficient and DMU_E is weakly efficient, while according to *Table 3*, DMU_E rank is better than DMU_A rank. Also, by to the results in *Table 3*, TDCE value of DMU_D during I_0 is greater than TDCE value of DMU_B. However, in

the time interval [0, 1/3], their ranking be opposite. Since during this time interval, TDCE value of DMU_B is equal to one, but TDCE value of DMU_D is less than one. So the ranking of DMUs in terms of TDCE would be changed if the time intervals were changed.

In this example, the input prices for all DMUs were assumed to be the same, although most of the time this is not the case. Since the DMUs offer their inputs from different malls and at different prices, the input prices may not be the same for all DMUs. This case is examined in the following example.

Example 2. Suppose that there are five DMUs with two inputs and one output. The inputs values and time dependent input prices for these DMUs are presented in *Table 4*. Moreover, assume thate output of all DMUs is the same value as $y = 1$.

Table 4. The input values and time dependent input prices for 5 DMUs.

DMU	Inputs		Time Dependent Input Prices	
	Input 1	Input 2	Input Price 1	Input Price 2
A	5	1	$9t^2 + 2$	$2t^2 + 3$
B	3	2	$1 + 6t$	$1 + 2t$
C	4	5	$t^2 + 1$	$2t^2 + 1$
D	2	4	$1 + 5t$	$1 + t$
E	2	6	$7t + 2$	$3t + 3$

Model (4) is applied to the data in *Table 4* and the following result is obtained:

$$C_A^*(t) = \begin{cases} 31t^2 + 12, & 0 \le t \le 2\sqrt{5}/5, \\ 26t^2 + 16, & t > 2\sqrt{5}/5. \end{cases}$$

$$C_B^*(t) = \begin{cases} 22t + 5, & 0 \le t \le 1/2, \\ 20t + 6, & t > 1/2. \end{cases}$$

$$C_C^*(t) = 7t^2 + 5,$$

$$C_D^*(t) = \begin{cases} 17t + 5, & 0 \le t \le 1/3, \\ 14t + 6, & t > 1/3. \end{cases}$$

$$C_E^*(t) = \begin{cases} 27t + 12, & 0 \le t \le 4, \\ 26t + 16, & t > 4. \end{cases}$$

By considering the inputs of each DMU and their cost and using *Eq. (3)*, the observed cost by each DMU is obtained as follows:

$$C_A(t) = 47t^2 + 13,$$

$$C_B(t) = 22t + 5,$$

$$C_C(t) = 14t^2 + 9,$$

$$C_D(t) = 14t + 6,$$

$$C_E(t) = 32t + 22.$$

By applying *Eq. (5)* for each *DMUs*, the TDCE function of each *DMU* is as following:

$$CE_A(t) = \begin{cases} \dfrac{31t^2 + 12}{47t^2 + 13}, & 0 \le t \le 2\sqrt{5}/5, \\[2mm] \dfrac{26t^2 + 16}{47t^2 + 13}, & t > 2\sqrt{5}/5. \end{cases}$$

$$CE_B(t) = \begin{cases} 1, & 0 \le t \le 1/2, \\[2mm] \dfrac{20t + 6}{22t + 5}, & t > 1/2. \end{cases}$$

$$CE_C(t) = \dfrac{7t^2 + 5}{14t^2 + 9},$$

$$CE_D(t) = \begin{cases} \dfrac{17t + 5}{14t + 6}, & 0 \le t \le 1/3, \\[2mm] 1, & t > 1/3. \end{cases}$$

$$CE_E(t) = \begin{cases} \dfrac{27t + 12}{32t + 22}, & 0 \le t \le 4, \\[2mm] \dfrac{26t + 16}{32t + 22}, & t > 4. \end{cases}$$

According to the above functions, none of DMUs are global-efficient. B and D are local-efficient and A, C and E are none-efficient. However, if we take the time interval [0, 1/2], B will be global-efficient. It is good to mention that in this example, similar to example 1, there is at least one DMU at each time whose cost efficiency value is equal to 1. In other words, if $0 \le t < 1/3$, $CE_B(t) = 1$ and B is cost efficient, if $1/3 \le t \le 1/2$, $CE_B(t) = 1$ and $CE_D(t) = 1$ and if $1/2 < t$, $CE_D(t) = 1$.

Suppose DMUs are evaluated in the interval $I_0 = [0,1]$. In this case, TDCE value of A, C, D, and E are unchanged compared to the previous example, and $CE_B^{I_0} = .9891$. These values are shown in *Table 5*.

Table 5. The TDCE value of DMUs over time interval I_0.

DMU	A	B	C	D	E
$CE_k^{I_0}$, $k = A,B,C,D,E$	0.8093	0.9891	0.5399	0.9775	0.6596

According to *Table 5*, DMUs were fully ranked. The ranking of these *DMUs* is as follows: B, D, A, E, C.

By comparing *Tables* 3 and *5*, when the input prices are changed, the rankings of the *DMUs* may also change. In Example 1, DMU_A initially was ranked fourth, but it was ranked third when input prices were considered differentlly for DMUs. In addition, the ranking of DMU_B and DMU_D also changed compared to Example 1.

In the previous two examples, we assumed the output of *DMUs* to be one, but in real world, the outputs cannot always be the same. In the next example, we will examine this case.

Example 3. Suppose that there are five DMUs with two inputs and one output. The inputs and output values and time dependent input prices for these DMUs are presented in *Table 6*.

Table 6. The input and output values and time dependent input prices for 5 DMUs.

DMU	Inputs		Output	Time Dependent Input Prices	
	Input 1	Input 2		Input Price 1	Input Price 2
A	5	1	2	$9t^2 + 2$	$2t^2 + 3$
B	3	2	3	$1 + 6t$	$1 + 2t$
C	4	5	1	$t^2 + 1$	$2t^2 + 1$
D	2	4	2	$1 + 5t$	$1 + t$
E	2	6	2	$7t + 2$	$3t + 3$

Model (4) is applied to the data in *Table 6* and the following result is obtained:

$$C_A^*(t) = \begin{cases} 31t^2 + 12, & 0 \le t \le 2\sqrt{5}/5, \\ 26t^2 + 16, & t > 2\sqrt{5}/5. \end{cases}$$

$$C_B^*(t) = 22t + 5,$$

$$C_C^*(t) = 7t^2 + 5,$$

$$C_D^*(t) = \begin{cases} 17t + 5, & 0 \le t \le 1/3, \\ 14t + 6, & t > 1/3. \end{cases}$$

$$C_E^*(t) = \begin{cases} 27t + 12, & 0 \le t \le 4, \\ 26t + 16, & t > 4. \end{cases}$$

By considering the inputs of each DMU and their costs and using *Eq. (3)*, the observed cost by each DMU is obtained as follows:

$$C_A(t) = 47t^2 + 13,$$

$$C_B(t) = 22t + 5,$$

$$C_C(t) = 14t^2 + 9,$$

$$C_D(t) = 14t + 6,$$

$$C_E(t) = 32t + 22.$$

By applying *Eq. (5)* for each DMUs, TDCE function of each DMU is as follows:

$$CE_A(t) = \begin{cases} \dfrac{31t^2 + 12}{47t^2 + 13}, & 0 \le t \le 2\sqrt{5}/5, \\[2mm] \dfrac{26t^2 + 16}{47t^2 + 13}, & t > 2\sqrt{5}/5. \end{cases}$$

$$CE_B(t) = 1,$$

$$CE_C(t) = \dfrac{7t^2 + 5}{14t^2 + 9},$$

$$CE_D(t) = \begin{cases} \dfrac{17t + 5}{14t + 6}, & 0 \le t \le 1/3, \\[2mm] 1, & t > 1/3. \end{cases}$$

$$CE_E(t) = \begin{cases} \dfrac{27t + 12}{32t + 22}, & 0 \le t \le 4, \\[2mm] \dfrac{26t + 16}{32t + 22}, & t > 4. \end{cases}$$

According to the above functions, in each interval under evaluation, B is global-efficient, D is local-efficient and A, C and E are none-efficient.

Suppose DMUs are evaluated in interval $I_0 = [0,1]$. In this case, TDCE value of DMUs in time interval I_0 is calculated by applying the *Eq. (8)* to 4 decimal places and the following results are obtained:

Table 7. The TDCE value of DMUs over time interval I_0.

DMU	A	B	C	D	E
$CE_k^{I_0}$, $k = A,B,C,D,E$	0.8093	1.0000	0.5399	0.9775	0.6596

According to *Table 7*, DMUs were fully ranked. The ranking of these DMUs is as follows: B, D, A, E, C. In this example, B is global, efficient in each interval under evaluation. This point did not exist in previous examples.

As these examples illustrate, the recommended method can be used in a number of ways. If the input prices are time-dependent and the same for all inputs or different for inputs, the proposed model is applicable. When evaluating DMUs, given time-dependent input prices, at any point in time there is at least one DMU whose cost-efficiency value is equal to 1. However, here may not be a DMU whose TDCE value in the interval under evaluation is equal to one. In another word, there may not be a globally efficient DMU, but there is at least one locally efficient DMU. In addition, this method can be a relatively powerful tool for classifying DMUs since, according to the formula defined for TDCE, it is very unlikely that the value of TDCE will be the same for DMUs.

4 | Conclusion

The study of DMUs provides a number of results, one of which is the cost efficiency of DMUs that introduced by DEA models. Cost efficiency evaluates a DMU's ability to produce current outputs at the lowest possible cost. In traditional cost efficiency models, the cost efficiency of the DMUs is calculated in a fixed time. Therefore, the prices are considered fixed and certain. While for most practical problems, input prices fluctuate over time. In other words, input prices are time dependent. In this study, a new model was introduced to calculate cost efficiency in the presence of time dependent input prices. In fact, in the model presented in this study, the inputs and outputs were considered a function of time. This feature is the superiority of this model over previous models. This model is a parametric programming problem and was solved with a parametric programming problem algorithm. Furthermore, the definitions of TDCE were provided. Finally, in the examples, the TDCE of DMUs was measured over a given time interval, and the DMUs were ranked. The researchers strongly recommend further research in the following areas. First, in this study, a numerical example was used to verify the proposed model. Other researchers can use this model to solve real problems. Second, the TDCE model proposed in this study was based on that proposed by Färe et al. [9], that introduced cost efficiency model. TDCE for DMUs can also be calculated based on Tone's cost efficiency model. Third, the model proposed in this study was in VRS mode. A model for constant return to scale (CRS model) can also be proposed.

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Conflicts of Interest

All co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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