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An inverse network DEA model for two-stage processes in the presence of undesirable factors

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Abstract

The inverse data envelopment analysis (DEA) problem has been one of the most important issues in the last decade. The inverse DEA permits the chief manager to increase (or decrease) outputs (or inputs) of decision-making units (DMUs) in such a way that the level of the relative efficiency of the under–observed DMU is preserved. Due to the importance of network-structured production systems in real life, the main purpose of the present research is to provide an inverse DEA model for a two-stage network-structured production system in the presence of undesirable factors. The weak disposability assumption is used to handle undesirable outputs in the proposed model. The focus of the proposed model is on estimating the amount of change in one or more indicators of one stage of the process by changing the indicators of another stage to preserve the level of efficiency. The most important advantage of the proposed procedure is that it can increase the level of outputs and simultaneously reduce the level of inputs. To demonstrate its practical use, the model is applied to a real-life example in poultry farming.

Keywords: Inverse data envelopment analysis, Efficiency, Two-stage network system, Undesirable factors, Weak disposability.

1 | Introduction

The concept of inverse DEA was first introduced by Zhang and Liu in [1]. Wei et al. [2] proposed a multi-objective linear programming approach in 2000. They considered a DEA model with three binary parameters, determined by constant, variable, or non-increasing/non-decreasing returns to scale assumptions, to evaluate the relative efficiency of decision-making units (DMUs). The proposed model explored the relationship between inputs and outputs for a specific unit, investigating the necessary changes in inputs or outputs to maintain the unit's efficiency level. The authors transformed the MOLP model into a single objective linear programming problem by using a weighted sum as the objective function.

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Yan et al. [3] introduced an inverse DEA model with preference cones. Jahanshahloo et al. [4, 5] further extended this model by proposing methods to estimate output levels while increasing inputs and improving efficiency and to determine input levels while decreasing outputs and improving efficiency. Jahanshahloo et al. [6] also proposed a modified inverse DEA model for sensitivity analysis of efficient and inefficient units. Hadi-vencheh and Foroughi [7] proposed a different inverse DEA model that increases or decreases inputs and outputs simultaneously. Alinezhad et al. [8] used an interactive MOLP to solve inverse DEA problems. Letworasirikul et al. [9] discussed an inverse DEA model with variable returns to scale assumption (on the BCC model). Li and Cui [10,11] utilized inverse DEA to determine returns to scale and scale elasticity for DMUs, with the efficiency value subject to change in their proposed model. Ghobadi and Jahangiri [12] used a technique for using fuzzy data in the inverse DEA model, while Gattoufi et al. [13] introduced a technique based on inverse DEA to suggest input or output levels to achieve a specific efficiency level. Jahanshahloo et al. [14] introduced an inverse DEA model under the inter-temporal dependence assumption, and Eyni et al. [15] applied inverse DEA to DMUs with undesirable inputs and outputs. Amin et al. [16,17] studied the effect of merging two decision-making units on the efficiency frontier using inverse DEA and used the same model to determine minimum performance levels. Also, Emrouznejad et al. [18] introduced an inverse DEA model to solve the problem of allocating the CO_2 emission for Chines manufacturing industries. Ghiyasi and Zhu [19] introduced an inverse DEA model for dealing with negative data in real-world applications. Lim [20] proposed an inverse DEA model for operational planning, considering frontier changes, and applied it to evaluate the Korean natural gas industry.

Considering this fact that many of the manufacturing processes in the real world, have a network structure, and internal structure affects the performance of units, Färe and Grosskopf [21] proposed Network DEA (NDEA) model. In the last two decades, extensive studies have been done on this subject. This includes works by Seiford and Zhu [22], Kao and Hwang [23], Liang et al. [24], Chen et al. [25], and Tone and Tsutsui [26], who applied DEA models to two-stage processes. Additionally, Zha and Liang [27], Chen et al. [28], and Amirteimoori et al. [29] conducted studies on two-stage processes with shared inputs. Later, researchers such as Fukuyama and Weber [30], Lozano et al. [31], Maghbouli et al. [32], Amirteimoori et al. [33], Wu et al. [34], and Nematizadeh and Nematizadeh [35] proposed models for two-stage processes that include undesirable factors.

Kalantary et al. [36] proposed a model called the inverse network dynamic Range Adjusted Model (RAM) to assess the sustainability of supply chains. The model changes both inputs and outputs in such a way that the efficiency remains unchanged, but it must not exceed the defined limits of inputs and outputs. After that, Kalantary and Farzipoor [37] developed an inverse network dynamic SBM model to evaluate the sustainability of the supply chain. This model changes the amount of input and output and produces new values. Amin and Ibn Boamah [38] created two-stage inverse DEA model to estimate potential gains from mergers, and Moghaddas et al. [39] developed inverse DEA models to enhance the sustainability of supply chain performance with a network series structure. Arbabi et al. [40] used a leader-follower method and presented an inverse DEA model for a two-stage production process.

The importance of two-stage network structures in practical applications, and the analysis of how changes in inputs or outputs impact efficiency in such a way that the level of efficiency is unchanged, has motivated the creation of an inverse model for two-stage processes. This study aims to present an inverse model for two-stage production structures that take into account undesirable outputs. The proposed approach has two advantages: first, it uses the weak disposability assumption to consider both desirable and undesirable factors, and second, it is formulated under variable returns to scale condition. The primary objective of this model is to increase and decrease outputs (or inputs) simultaneously and calculate the impact on other indicators (input or output) in such a way that the efficiency level of the units remains unchanged.

The rest of the paper is organized as follows: Section 2 briefly reviews the inverse DEA model and weak disposability assumption. In Section 3, we propose an inverse DEA model for two-stage processes with undesirable factors. A case study is mentioned to analyze the suggested approach in Section 4. Finally, in Section 5, conclusions are given.

2| Related works

In this section, first, a brief review of the inverse DEA model, introduced by Wei et al. [2], is given. Then, the weak disposability assumption by Shephard [41] is explained.

2.1| Inverse DEA model

Assume that there are K DMUs ($DMU_k: k=1,...,K$) and each DMU_k uses inputs $x_{ik}\,(i=1,...,m,\ k=1,..,K)$ to produce outputs $y_{rk}\,(r=1,...,s,\ k=1,...,K)$. Yu et al. [42,43] introduced the following generalized output-oriented DEA model by considering three binary parameters δ_1, δ_2 and δ_3 to a general model:

$$\begin{aligned}
&\varphi_o^* = Max\ \varphi \\
&s.t. \\
&\sum_{k=1}^{K} \lambda_k x_{ik} \le x_{io},\ i=1,...,m, \\
&\sum_{k=1}^{K} \lambda_k y_{rk} \ge \varphi\, y_{ro},\ r=1,...,s, \\
&\delta_1(\sum_{k=1}^{K} \lambda_k + \delta_2(-1)^{\delta_3}\nu) = \delta_1, \\
&\nu \ge 0,\ \lambda_k \ge 0,\ k=1,...,K.
\end{aligned} \tag{1}$$

in which φ_o^* is the optimal efficiency value of DMU_o. If $\varphi_o^*=1$, DMU_o is weakly efficient. Also, if $\varphi_o^*>1$, DMU_o is inefficient.

Then, Wei et al. [2] supposed that the inputs of DMU_o are increased from x_o to $\alpha_o = x_o + \Delta x_o$ with $\Delta x_o \ge 0$ and $\Delta x_o \ne 0$. The aim in their model was estimating the output vector β_o, that $\beta_o = (\beta_{1o}, \beta_{2o},...,\beta_{so})^T = y_o + \Delta y_o$ and $\Delta y_o \ge 0$, in such a way that the efficiency value of DMU_o is still φ_o^*. Suppose DMU_{k+1} represents the new DMU_o after the changes of inputs and outputs. The efficiency value of DMU_{k+1} is obtained by solving the following model:

$$\begin{aligned}
&\varphi_{K+1}^* = Max\ \varphi \\
&s.t. \\
&\sum_{k=1}^{K} \lambda_k x_{ik} + \lambda_{K+1}\alpha_{io} \le x_{io},\ i=1,...,m, \\
&\sum_{k=1}^{K} \lambda_k y_{rk} + \lambda_{K+1}\beta_{ro} \ge \varphi\beta_{ro},\ r=1,...,s, \\
&\delta_1(\sum_{k=1}^{K} \lambda_k + \lambda_{K+1} + \delta_2(-1)^{\delta_3}\nu) = \delta_1, \\
&\nu \ge 0,\ \lambda_k \ge 0,\ k=1,...,K,K+1.
\end{aligned} \tag{2}$$

If the optimal values of *Eq. (1)* and *Eq. (2)* are equal, we can say that the efficiency score of DMU_o remains unchanged. Moreover, Wei et al. [2] introduced the following linear multi-objective programming problem:

$$
\begin{aligned}
& Max\ (\beta_{1o}, \beta_{2o}, ..., \beta_{so}) \\
& s.t. \\
& \sum_{k=1}^{K} \lambda_k x_{ik} \le \alpha_{io},\ i = 1,...,m, \\
& \sum_{k=1}^{K} \lambda_k y_{rk} \ge \varphi_o^* \beta_{ro},\ r = 1,...,s, \\
& \delta_1 (\sum_{k=1}^{K} \lambda_k + \delta_2 (-1)^{\delta_3} \nu) = \delta_1, \\
& \beta_{ro} \ge y_{ro},\ r = 1,...,s, \\
& \nu \ge 0,\ \lambda_k \ge 0,\ k = 1,...,K.
\end{aligned}
\tag{3}
$$

In *Eq (3)*, φ_o^* is the optimal objective value of *Eq. (1)*, and also α_o is predefined. Note that *Eq. (3)* is a MOLP problem and by changing the objective function as a weighted sum of β_r, the problem is transformed into a linear programming problem. *Eq. (3)* can determine the amount of outputs changes while the efficiency values of the DMUs remain unchanged.

2.2| Weak disposability assumption

Assume that we have K DMUs, and each $DMU_k\ (k = 1,...,K)$ consists of input vector $x_k = (x_{1k},...,x_{mk}) \ge 0$, desirable and undesirable output vectors $v_k = (v_{1k},...,v_{sk}) \ge 0$ and $w_k = (w_{1k},...,w_{tk}) \ge 0$, respectively. In general form, the production possibility set is defined as

$$T = \{(x, v, w) : (v, w) \text{ can be produced by } x\}$$

Shephard [41] defined the weak disposability assumption as follows:

Definition 1. Outputs (v, w) are weakly disposable if and only if $(x, v, w) \in T$ and $\theta \in [0,1]$ implies $(x, \theta v, \theta w) \in T$.

Based on this definition, Färe and Grosskopf [44] have used the weak disposability assumption to introduce the following technology set:

$$T=\{(x,v,w):$$
$$\sum_{k=1}^{K}\lambda_k x_{ik}\le x_{io},\ i=1,\dots,m,$$
$$\sum_{k=1}^{K}\theta\lambda_k v_{rk}\ge v_{ro},\ r=1,\dots,s,$$
$$\sum_{k=1}^{K}\theta\lambda_k w_{tk}=w_{to},\ t=1,\dots,l, \qquad (4)$$
$$\sum_{k=1}^{K}\lambda_k=1,$$
$$0\le\theta\le1,\ \lambda_k\ge0,\ k=1,\dots,K\}.$$

Färe and Grosskopf [44] considered a single abatement factor for all DMUs, and in a modified approach, Kuosmanen [45] used different abatement factor θ_k for each DMU_k $(k=1,\dots,K)$, and he improved the above technology set in the following linear form:

$$T_{New}=\{(x,v,w):$$
$$\sum_{k=1}^{K}(\mu_k+\gamma_k)\,x_{ik}\le x_{io},\ i=1,\dots,m,$$
$$\sum_{k=1}^{K}\mu_k v_{rk}\ge v_{ro},\ r=1,\dots,s,$$
$$\sum_{k=1}^{K}\mu_k w_{tk}=w_{to},\ t=1,\dots,l, \qquad (5)$$
$$\sum_{k=1}^{K}(\mu_k+\gamma_k)=1,$$
$$\mu_k,\ \gamma_k\ge0,\ k=1,\dots,K\}.$$

in which $\mu_k=\theta_k\lambda_k$ and $\gamma_k=(1-\theta_k)\lambda_k$ $(k=1,\dots,K)$.

Kuosmanen [45] correctly claimed that this is the correct technology set when we use the weak disposability assumption to handle undesirable outputs.

3| The proposed inverse DEA model

The main contribution of this section is to introduce an inverse DEA model for a two-stage network structure including undesirable outputs in such a way that it is able to analyze the effect of final output changes on inputs and outputs of both stages by assuming the overall efficiency remains unchanged. This method allows outputs to be increased and decreased, simultaneously. The following algorithm shows the summary of our proposed approach:

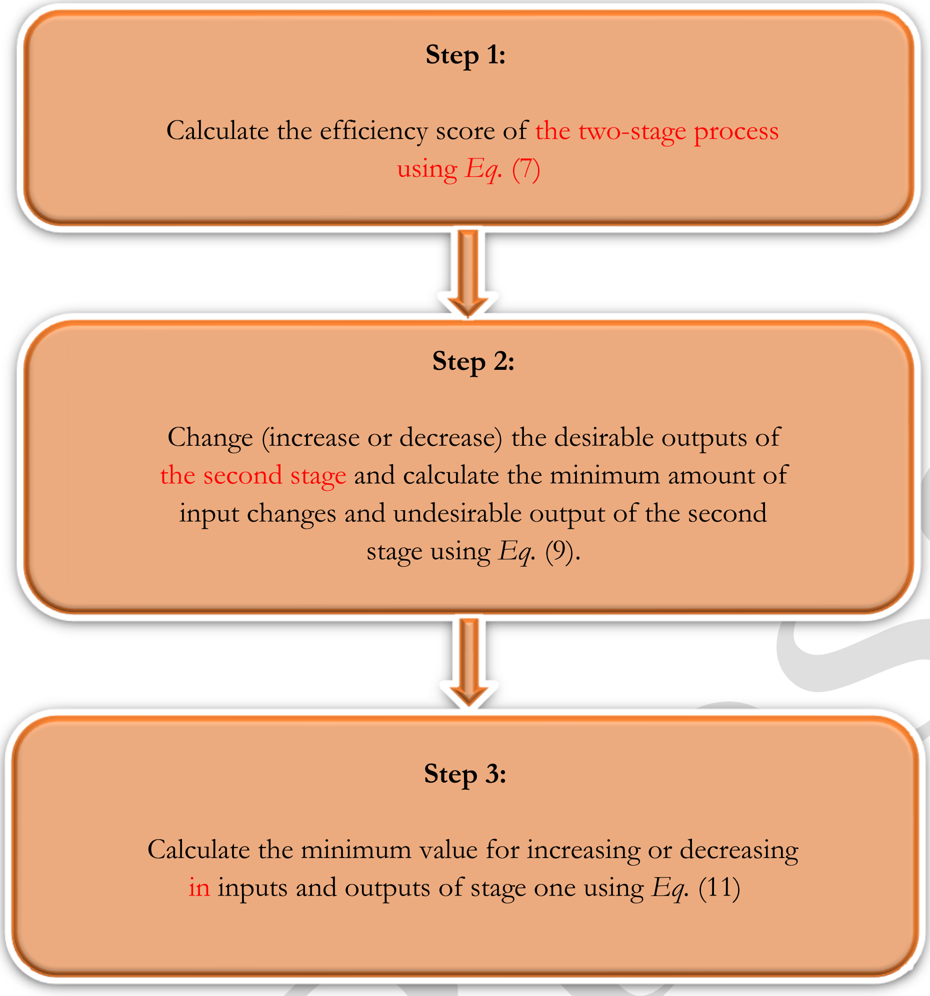

Fig. 1. The proposed approach algorithm.

Now, we will describe the proposed method in detail. First, consider a two-stage process with undesirable outputs as *Fig. 2*.

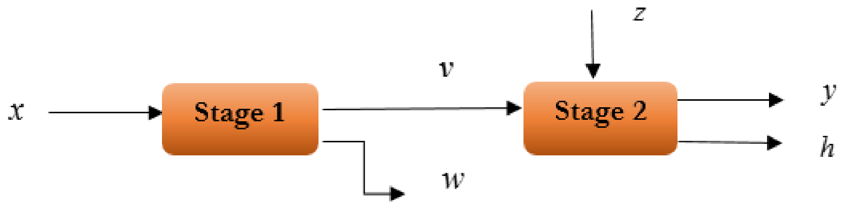

Fig. 2. Two-stage feedback process of DMU_k.

Suppose that there are K DMUs and each DMU_k $(k=1,...,K)$ consists of two stages. Stage one consumes the input vector $x_k=(x_{1k},...,x_{Nk})\geq 0$ to produce desirable and undesirable output vectors $v_k=(v_{1k},...,v_{Mk})\geq 0$ and $w_k=(w_{1k},...,w_{Jk})\geq 0$, respectively. Stage 2 is fed by the desirable output v_k and the input vector $z_k=(z_{1k},...,z_{Tk})\geq 0$. The final outputs from stage 2 are the desirable and undesirable outputs $y_k=(y_{1k},...,y_{Rk})\geq 0$ and $h_k=(h_{1k},...,h_{Ik})\geq 0$, respectively.

The linear technology set under the variable returns to scale and weak disposability assumption for the above-mentioned two-stage process is defined as follows:

$$
\begin{aligned}
&T=\{(x,v,w,z,y,h):\\
&\text{Stage 1 constraints:}\\
&\sum_{k=1}^{K}(\rho_k+\mu_k)\,x_{nk}\le x_{no},\ n=1,...,N,\\
&\sum_{k=1}^{K}\rho_k v_{mk}+s_m=v_{mo},\ m=1,...,M,\\
&\sum_{k=1}^{K}\rho_k w_{jk}=w_{jo},\ j=1,...,J,\\
&\text{Stage 2 constraints:}\\
&\sum_{k=1}^{K}\rho_k v_{mk}+s_m=v_{mo},\ m=1,...,M,\\
&\sum_{k=1}^{K}(\rho_k+\mu_k)\,z_{tk}\le z_{to},\ t=1,...,T,\\
&\sum_{k=1}^{K}\rho_k y_{rk}\ge y_{ro},\ r=1,...,R,\\
&\sum_{k=1}^{K}\rho_k h_{ik}=h_{io},\ i=1,...,I,\\
&\text{Generic constraints:}\\
&\sum_{k=1}^{K}(\rho_k+\mu_k)=1,\\
&\rho_k,\ \mu_k\ge 0,\ k=1,...,K,\\
&s_m\ \text{are free for}\ m=1,...,M\}.
\end{aligned}
\tag{6}
$$

Note that the intermediate vector $v_k=(v_{1k},...,v_{Mk})$ plays two roles, it is output from stage one and at the same time, it is input for stage two. In this sense, we considered a free slack variable s_m $(m=1,...,M)$ to the corresponding constraint in the technology set (6).

Taking the technology set (6) into consideration, the following model is proposed to evaluate the relative efficiency of a specific DMU_o:

$$
\begin{aligned}
& e_o^* = Min\ \tfrac{1}{2}[\tfrac{1}{N+J}(\sum_{n=1}^{N}\alpha_n + \sum_{j=1}^{J}\beta_j) + \tfrac{1}{T+I}(\sum_{t=1}^{T}\gamma_t + \sum_{i=1}^{I}\varphi_i)] \\
& s.t. \\
& \text{Stage 1 constraints:} \\
& \sum_{k=1}^{K}(\rho_k + \mu_k)\, x_{nk} \le \alpha_n x_{no},\ n = 1,\dots,N, \\
& \sum_{k=1}^{K}\rho_k v_{mk} + s_m = v_{mo},\ m = 1,\dots,M, \\
& \sum_{k=1}^{K}\rho_k w_{jk} = \beta_j w_{jo},\ j = 1,\dots,J, \\
& \text{Stage 2 constraints:} \\
& \sum_{k=1}^{K}\rho_k v_{mk} + s_m = v_{mo},\ m = 1,\dots,M, \\
& \sum_{k=1}^{K}(\rho_k + \mu_k)\, z_{tk} \le \gamma_t z_{to},\ t = 1,\dots,T, \\
& \sum_{k=1}^{K}\rho_k y_{rk} \ge y_{ro},\ r = 1,\dots,R, \\
& \sum_{k=1}^{K}\rho_k h_{ik} = \varphi_i h_{io},\ i = 1,\dots,I, \\
& \text{Generic constraints:} \\
& \sum_{k=1}^{K}(\rho_k + \mu_k) = 1, \\
& \rho_k,\ \mu_k \ge 0,\ k = 1,\dots,K, \\
& s_m \text{ are free for } m = 1,\dots,M, \\
& 0 \le \alpha_n,\ \beta_j,\ \gamma_t,\ \varphi_i \le 1\ \forall n, j, t, i.
\end{aligned} \tag{7}
$$

Definition 2. DMU_o is overall efficient if and only if $e_o^* = 1$.

Clearly, DMU_o is overall efficient if and only if both stages are efficient.

Definition 3. The first stage efficiency of DMU_o is defined as $e_o^{*Stage1} = \frac{1}{N+J}(\sum_{n=1}^{N}\alpha_n^* + \sum_{j=1}^{J}\beta_j^*)$ and it is said to be efficient if and only if $e_o^{*Stage1} = 1$, in which $\alpha_n^*,\ \beta_j^*\ (\forall n, j)$ are optimal values obtained from *Eq. (7)*.

Definition 4. The second stage efficiency of DMU_o is defined as $e_o^{*Stage2} = \frac{1}{T+I}(\sum_{t=1}^{T}\gamma_t^* + \sum_{i=1}^{I}\varphi_i^*)$ and it is said to be efficient if and only if $e_o^{*Stage1} = 1$, in which $\gamma_t^*,\ \varphi_i^*\ (\forall t, i)$ are optimal values obtained from *Eq. (7)*.

Now, we suggest an inverse DEA model for the process given in *Fig.* 2. It is to be noted that any change in one stage effects the other stage. Therefore, we first change the desirable output of stage two from y_o to $y_o + \Delta y_o$, which Δy_o can be positive or negative. Then, we calculate the amount of inputs changes and undesirable outputs of the second stage by using the following model (See *Fig.* 3):

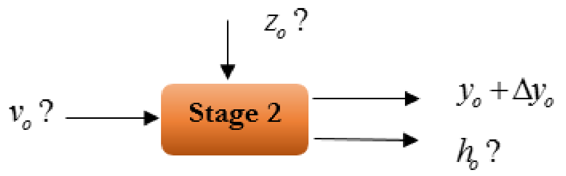

Fig. 3. The second stage of DMU_o.

$$
\begin{aligned}
& e_o^{*InvStage2} = Min\ [\sum_{m=1}^{M} c_m^{(v)} |\Delta v_m| + \sum_{t=1}^{T} c_t^{(z)} |\Delta z_t| + \sum_{i=1}^{I} c_i^{(h)} |\Delta h_i|] \\
& s.t. \\
& \sum_{k=1}^{K} \rho_k v_{mk} = v_{mo} + \Delta v_{mo},\ m = 1,...,M, \\
& \sum_{k=1}^{K} (\rho_k + \mu_k)\ z_{tk} \le \gamma_o^* (z_{to} + \Delta z_{to}),\ t = 1,...,T, \\
& \sum_{k=1}^{K} \rho_k y_{rk} \ge y_{ro} + \Delta y_{ro},\ r = 1,...,R, \\
& \sum_{k=1}^{K} \rho_k h_{ik} = \varphi_o^* (h_{io} + \Delta h_{io}),\ i = 1,...,I, \\
& \sum_{k=1}^{K} (\rho_k + \mu_k) = 1, \\
& \rho_k,\ \mu_k \ge 0,\ k = 1,...,K, \\
& \Delta v_m,\ \Delta z_t,\ \Delta h_i \ \ are\ free\ \forall m,t,i.
\end{aligned}
\qquad (8)
$$

in which, Δv_m, Δz_t and Δh_i are amounts of increasing or decreasing of inputs and outputs. Also, $c_m^{(v)}$, $c_t^{(z)}$, $c_i^{(h)}$ $(\forall m,i,t)$ are weights assigned to inputs and outputs, and γ_o^*, φ_o^* are the optimal objective values obtained from *Eq. (7)*. Clearly, *Eq. (8)* is non-linear and it can be transformed into the following linear form:

$$
\begin{aligned}
& e_o^{*InvStage2} = Min \; [\sum_{m=1}^{M} c_m^{(v)} (\Delta v_m^+ + \Delta v_m^-) + \sum_{t=1}^{T} c_t^{(z)} (\Delta z_t^+ + \Delta z_t^-) + \sum_{i=1}^{I} c_i^{(h)} (\Delta h_i^+ + \Delta h_i^-)] \\
& s.t. \\
& \sum_{k=1}^{K} \rho_k v_{mk} = v_{mo} + (\Delta v_m^+ - \Delta v_m^-), \; m = 1,..., M, \\
& \sum_{k=1}^{K} (\rho_k + \mu_k) \; z_{tk} \le \gamma_o^* \; [z_{to} + (\Delta z_t^+ - \Delta z_t^-)], \; t = 1,..., T, \\
& \sum_{k=1}^{K} \rho_k y_{rk} \ge y_{ro} + \Delta y_{ro}, \; r = 1,..., R, \\
& \sum_{k=1}^{K} \rho_k h_{ik} = \varphi_o^* \; [h_{io} + (\Delta h_i^+ - \Delta h_i^-)], \; i = 1,..., I, \\
& \sum_{k=1}^{K} (\rho_k + \mu_k) = 1, \\
& \rho_k, \; \mu_k \ge 0, \; k = 1,..., K, \\
& \Delta v_m^+, \; \Delta v_m^-, \; \Delta z_t^+, \; \Delta z_t^-, \; \Delta h_i^+, \; \Delta h_i^- \ge 0 \; \forall m, t, i.
\end{aligned} \tag{9}
$$

Note that $e_o^{*InvStage2}$ is not the efficiency value for each DMU, but it is the inverse value and it shows the minimum value of increase or decrease for inputs and outputs.

After calculating the inputs and outputs changes of stage two, we consider the following model to calculate the minimum value for increasing or decreasing in the inputs and outputs of stage one (See *Fig.* 4):

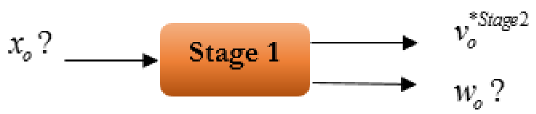

Fig. 4. The first stage of DMU_o.

$$
\begin{aligned}
& e_o^{*InvStage1} = Min \; [\sum_{n=1}^{N} c_n^{(x)} \left|\Delta x_n\right| + \sum_{j=1}^{J} c_j^{(w)} \left|\Delta w_j\right|] \\
& s.t. \\
& \sum_{k=1}^{K} (\rho_k + \mu_k) \; x_{nk} \le \alpha_o^* (x_{no} + \Delta x_{no}), \; n = 1,..., N, \\
& \sum_{k=1}^{K} \rho_k v_{mk} = v_{mo} + \Delta v_{mo}^{*Stage2}, \; m = 1,..., M, \\
& \sum_{k=1}^{K} \rho_k w_{jk} = \beta_o^* (w_{jo} + \Delta w_{jo}), \; i = 1,..., I, \\
& \sum_{k=1}^{K} (\rho_k + \mu_k) = 1, \\
& \rho_k, \; \mu_k \ge 0, \; k = 1,..., K, \\
& \Delta x_n, \; \Delta w_j \; are \; free \; \forall m, t, i.
\end{aligned} \tag{10}
$$

In *Eq. (10)*, $c_n^{(x)}$, $c_j^{(w)}$ $(\forall n, j)$ are weighted assigned for inputs and outputs. α_o^*, β_o^* are the optimal values obtained from *Eq. (7)* and $\Delta v_{mo}^{*Stage2}$ is the minimum change for the output v_m and it is obtained from *Eq. (9)*. Also, Δx_n, Δw_j $(\forall n, j)$ are amounts of increasing or decreasing changes of inputs and outputs. Note that, in this model $e_o^{*InvStage1}$ is not the efficiency value for stage one, but it is the minimum value for increasing or decreasing the inputs and outputs. *Eq. (10)* is non-linear and it can be transformed into the following linear form:

$$\begin{aligned}
& e_o^{*InvStage1} = Min\ [\sum_{n=1}^{N} c_n^{(x)}(\Delta x_n^+ + \Delta x_n^-) + \sum_{j=1}^{J} c_j^{(w)}(\Delta w_j^+ + \Delta w_j^-)] \\
& s.t. \\
& \sum_{k=1}^{K}(\rho_k + \mu_k)\, x_{nk} \le \alpha_o^*\ [x_{no} + (\Delta x_n^+ - \Delta x_n^-)],\ n = 1,\dots,N, \\
& \sum_{k=1}^{K}\rho_k v_{mk} + s_m = v_{mo} + \Delta v_{mo}^{*Stage2},\ m = 1,\dots,M, \\
& \sum_{k=1}^{K}\rho_k w_{jk} = \beta_o^*\ [w_{jo} + (\Delta w_j^+ - \Delta w_j^-)],\ j = 1,\dots,J, \\
& \sum_{k=1}^{K}(\rho_k + \mu_k) = 1, \\
& \rho_k,\ \mu_k \ge 0,\ k = 1,\dots,K, \\
& \Delta x_n^+,\ \Delta x_n^-,\ \Delta w_j^+,\ \Delta w_j^- \ge 0\ \forall n, j.
\end{aligned} \tag{11}$$

After calculating the values of inputs and outputs changes of both stages, we evaluate the relative efficiency for each unit under evaluation by the *Eq. (7)*. It is noteworthy that the efficiency value for all DMUs remain unchange. The following theorem describes it easily:

Theorem 1. Let e_o^* be the relative efficiency of DMU_o, and the final desirable output of DMU_o changes from y_o to $y_o + \Delta y_o$. If $(\rho, \mu, x_o + \Delta x_o, v_o + \Delta v_o, w_o + \Delta w_o, z_o + \Delta z_o, h_o + \Delta h_o)$ be an optimal solution of *Eq. (9)* and *(11)*, then the efficiency score of DMU_o after changing inputs and outputs is e_o^*.

Proof. Suppose that $(\rho, \mu, x_o + \Delta x_o, v_o + \Delta v_o, w_o + \Delta w_o, z_o + \Delta z_o, h_o + \Delta h_o)$ is an optimal solution of *Eq. (9)* and *(11)*. Consider the following model, in which DMU_{k+1} represents DMU_o after changing the inputs and outputs:

$$\bar{e}_o^* = Min\ \tfrac{1}{2}[\tfrac{1}{N+J}(\sum_{n=1}^{N}\bar{\alpha}_n + \sum_{j=1}^{J}\bar{\beta}_j) + \tfrac{1}{T+I}(\sum_{t=1}^{T}\bar{\gamma}_t + \sum_{i=1}^{I}\bar{\varphi}_i)]$$

$s.t.$

Stage 1 constraints:

$$\sum_{k=1}^{K}(\rho_k + \mu_k)\, x_{nk} + (\rho_{k+1} + \mu_{k+1})\,(x_{no} + \Delta x_{no}) \le \bar{\alpha}_n (x_{no} + \Delta x_{no}),\ n = 1,\dots,N,$$

$$\sum_{k=1}^{K}\rho_k v_{mk} + \rho_{k+1}(v_{mo} + \Delta v_{mo}) + s_m = v_{mo} + \Delta v_{mo},\ m = 1,\dots,M,$$

$$\sum_{k=1}^{K}\rho_k w_{jk} + \rho_{k+1}(w_{jo} + \Delta w_{jo}) = \bar{\beta}_j (w_{jo} + \Delta w_{jo}),\ j = 1,\dots,J,$$

Stage 2 constraints:

$$\sum_{k=1}^{K}\rho_k v_{mk} + \rho_{k+1}(v_{mo} + \Delta v_{mo}) + s_m = v_{mo} + \Delta v_{mo},\ m = 1,\dots,M, \quad (12)$$

$$\sum_{k=1}^{K}(\rho_k + \mu_k)\, z_{tk} + (\rho_{k+1} + \mu_{k+1})\,(\mathrm{z}_{to} + \Delta z_{to}) \le \bar{\gamma}_t (z_{to} + \Delta z_{to}),\ t = 1,\dots,T,$$

$$\sum_{k=1}^{K}\rho_k y_{rk} + \rho_{k+1}(y_{ro} + \Delta y_{ro}) \ge y_{ro} + \Delta y_{ro},\ r = 1,\dots,\mathrm{R},$$

$$\sum_{k=1}^{K}\rho_k h_{ik} + \rho_{k+1}(h_{io} + \Delta h_{io}) = \bar{\varphi}_i (h_{io} + \Delta h_{io}),\ i = 1,\dots,I,$$

Generic constraints:

$$\sum_{k=1}^{K}(\rho_k + \mu_k) + (\rho_{k+1} + \mu_{k+1}) = 1,$$

$$\rho_k,\ \mu_k \ge 0,\ k = 1,\dots,K,K+1,$$

$$s_m \ \textit{are free for}\ m = 1,\dots,M,$$

$$0 \le \bar{\alpha}_n,\ \bar{\beta}_j,\ \bar{\gamma}_t,\ \bar{\varphi}_i \le 1\ \forall n, j, t, i.$$

$$
\begin{aligned}
&\overline{e}_o^{\,*} = Min\ \tfrac{1}{2}\Big[\tfrac{1}{N+J}\Big(\sum_{n=1}^{N}\overline{\alpha}_n + \sum_{j=1}^{J}\overline{\beta}_j\Big) + \tfrac{1}{T+I}\Big(\sum_{t=1}^{T}\overline{\gamma}_t + \sum_{i=1}^{I}\overline{\varphi}_i\Big)\Big] \\
&s.t. \\
&\text{Stage 1 constraints:} \\
&\sum_{k=1}^{K}(\rho_k+\mu_k)\,x_{nk} \le [\overline{\alpha}_n - (\rho_{k+1}+\mu_{k+1})](x_{no}+\Delta x_{no}),\ n=1,\dots,N, \\
&\sum_{k=1}^{K}\rho_k v_{mk} + s_m = (1-\rho_{k+1})(v_{mo}+\Delta v_{mo}),\ m=1,\dots,M, \\
&\sum_{k=1}^{K}\rho_k w_{jk} = (\overline{\beta}_j - \rho_{k+1})(w_{jo}+\Delta w_{jo}),\ j=1,\dots,J, \\
&\text{Stage 2 constraints:} \\
&\sum_{k=1}^{K}\rho_k v_{mk} + s_m = (1-\rho_{k+1})(v_{mo}+\Delta v_{mo}),\ m=1,\dots,M, \qquad\qquad (13) \\
&\sum_{k=1}^{K}(\rho_k+\mu_k)\,z_{tk} \le [\overline{\gamma}_t - (\rho_{k+1}+\mu_{k+1})](z_{to}+\Delta z_{to}),\ t=1,\dots,T, \\
&\sum_{k=1}^{K}\rho_k y_{rk} \ge (1-\rho_{k+1})(y_{ro}+\Delta y_{ro}),\ r=1,\dots,R, \\
&\sum_{k=1}^{K}\rho_k h_{ik} = (\overline{\varphi}_i - \rho_{k+1})(h_{io}+\Delta h_{io}),\ i=1,\dots,I, \\
&\text{Generic constraints:} \\
&\sum_{k=1}^{K}(\rho_k+\mu_k) + (\rho_{k+1}+\mu_{k+1}) = 1, \\
&\rho_k,\ \mu_k \ge 0,\ k=1,\dots,K,K+1, \\
&s_m\ \textit{are free for}\ m=1,\dots,M, \\
&0 \le \overline{\alpha}_n,\ \overline{\beta}_j,\ \overline{\gamma}_t,\ \overline{\varphi}_i \le 1\ \forall n,j,t,i.
\end{aligned}
$$

Case 1: $e_o^* < 1$ [e_o^* is the efficiency score of *Eq. (7)*.]

If $\rho_{k+1} = 1$ and $\mu_{k+1} = 0$, then $\rho_k = \mu_k = 0\ (k=1,\dots,K)$, $\overline{\alpha}_n = \overline{\beta}_j = \overline{\gamma}_t = \overline{\varphi}_i = 1\ (\forall n,j,t,i)$ and $\overline{e}_o^{\,*} = 1$. $\overline{e}_o^{\,*} = 1$ is a feasible solution and it is not the optimal solution for *Eq. (7)* when $e_o^* < 1$, because we can obtain a better solution. If $\rho_{k+1} = \mu_{k+1} = 0$, constraints of *Eq.* (13) are transformed into constraints of *Eq. (7)*, then the optimal solution is e_o^*, with $e_o^* < 1$, i.e. a better solution is found for *Eq. (13)*. Therefore, the efficiency score of DMU_k $(k=1,\dots,K,K+1)$ will remain unchanged $(e_o^* = \overline{e}_o^{\,*})$.

Case 2: $e_o^* = 1$.

If $\rho_{k+1} = 1$ and $\mu_{k+1} = 0$, then $\rho_k = \mu_k = 0\ (k=1,\dots,K)$, $\overline{\alpha}_n = \overline{\beta}_j = \overline{\gamma}_t = \overline{\varphi}_i = 1\ (\forall n,j,t,i)$ and $\overline{e}_o^{\,*} = 1$, where $e_o^* = \overline{e}_o^{\,*} = 1$. Now, if $\rho_{k+1} = \mu_{k+1} = 0$, then the constraints of *Eq. (13)* are transformed into the constraints of *Eq. (7)*. Therefore, $e_o^* = \overline{e}_o^{\,*} = 1$.■

4 | Application

In this section, the real applicability of our proposed approach is shown by a real example on poultry farms. We consider 13 poultry farms, and each farm consists of two sections. Section one consumes new-born chickens and the other inputs to produce the 3-week chickens, and section two produces meat chickens and other outputs. *Fig.* 4 shows the structure of each poultry farm.

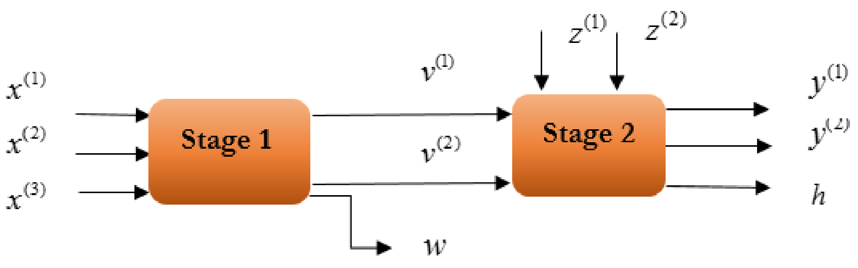

Fig 5. Structure of each poultry farm.

Details of inputs consumption and outputs production are illustrated as follows:

❖ **Stage 1:** 3-Week Chickens Production Section

- **Inputs:**

- New-Born Chickens (x^1): the number of newborn chickens.

- Nutrition Cost (x^2): costs needed to supply nutrition for chickens.

- Operational Expenses (x^3): costs needed to supply the staff wages, drugs, water, electricity and so on.

- **Desirable outputs:**

- Feed Conversion Ratio (v^1): FCR index is a ratio of food consumption to increase in weight of live chickens during a certain period of time.

- Produced Meat (v^2): weight of 21-days-old chicken.

- **Undesirable outputs:**

- Mortality (w): the number of chickens that dies.

❖ **Stage 2:** Meat Chickens Production Section

- **Inputs:**

- Nutrition Cost (z^1): cost of supplying nutrition for chickens.

- Operational Expenses (z^2): costs of supplying drugs, water, electricity and so on.

- **Desirable outputs:**

- Feed Conversion Ratio (y^1): FCR index is a ratio of food consumption to increase in weight of live chickens during a specified time.

- Produced Meat (y^2): weight of chicken meat produced at the end of the period.

- **Undesirable output:**

- Mortality (w): the number of chickens that dies.

In what follows, statistical summary of the inputs and outputs for 13 farms is compiled in *Table 1*.

Table 1. Statistical data of poultry farms.

Variables	Min	Max	Mean	Std. Dev.
x^1	11000	19800	13790	2059.791
x^2	133540	235970	165154.6	24670.04
x^3	51340	80960	62930.77	7092.317
v^1	1.62	1.75	1.68	0.037622
v^2	5633.8	10373.2	7223.292	1143.519
w	385	1263	692.6154	273.2859
z^1	378100	685800	474479.2	73066.27
z^2	86880	144430	110051.5	13676.94
y^1	1.88	2.04	1.98	0.044202
y^2	25683.4	44581.2	30810.84	4648.101
h	79	336	204.0769	74.10955

According to the data in *Tables 1 and 2*, profitability for each of 31 regions and their components are evaluated by *Eq. (10)*, *Eq. (7)* and *Eq. (11)*. The results are depicted in *Table 3*.

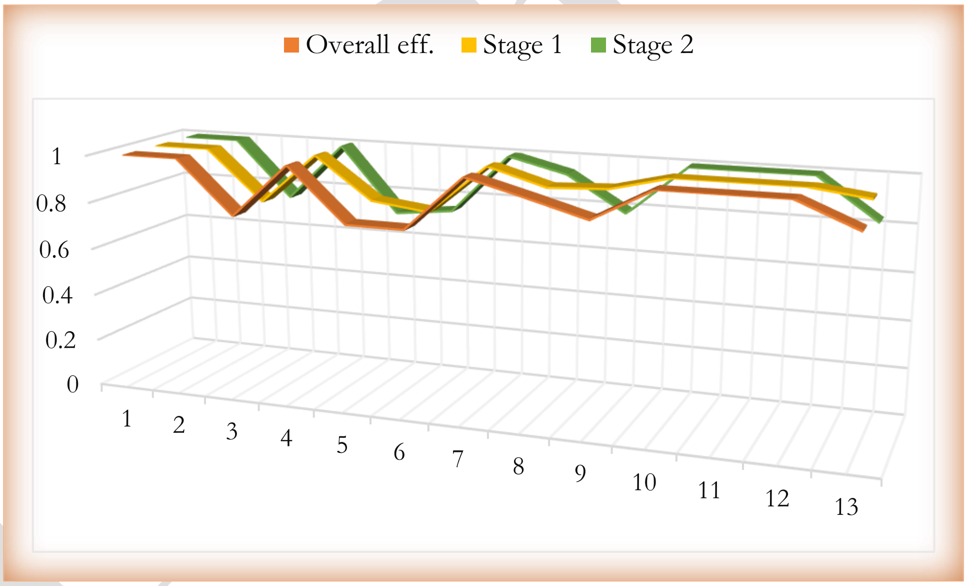

Fig 6. The results obtained from *Eq.* (7).

Table 2. Statistical data of poultry farms.

Poultry farms	$e_o^{*Overall}$	$e_o^{*Stage1}$	$e_o^{*Stage2}$
1	1.0000	1.0000	1.0000
2	1.0000	1.0000	1.0000
3	0.7731	0.7869	0.7593
4	1.0000	1.0000	1.0000
5	0.7735	0.8243	0.7227
6	0.7717	0.7938	0.7495

7	1.0000	1.0000	1.0000
8	0.9345	0.9292	0.9398
9	0.8664	0.9403	0.7926
10	1.0000	1.0000	1.0000
11	1.0000	1.0000	1.0000
12	1.0000	1.0000	1.0000
13	0.8979	0.9684	0.8274

In *Table 2*, the second column shows the efficiency score for the whole system and the other columns show the efficiency value for stage one and stage two of the system, respectively. The results in *Table 2* shows that the farms 1, 2, 4, 7, 10, 11 and 12 are overall efficient. Also, both sub-sections of these farms are also efficient. In order to further clarify the solving models in the previous section, we act in this way.

For convenience, suppose $c=(1,1,...,1)^T$. We studied some different cases separately. For example, in the first case, we increased one of the final desirable outputs of unit 3, that is inefficient. Indeed, y^2 is changed from 28506.3 to 28700. In the second case, we considered units 5 and 10, which one of them is inefficient and another one is efficient. For example, the desirable outputs of units 5 and 10, namely y^2, are changed from 26256.5 and 28223.5 to 26000 and 28500, respectively. We decreased outputs of one unit and increased outputs of the other one. Finally, in the third case, we again considered an inefficient unit 8, and increased its final desirable outputs, i.e. y^1 and y^2 are changed from 2.01 and 33414.6 to 2.04 and 36000, respectively.

By applying the above changes in the final desirable output of stage 2, the amount of increasing or decreasing of the other inputs and outputs are obtained from *Eq. (9)* and *(11)*. These results are listed in *Table 3*.

Table 3. Results obtained from inputs and outputs changing.

Unit	3	8	5	10
Δx^1	0	0	0	0
Δx^2	364.49	0	23.61	0
Δx^3	0	0	0	0
Δw	-91.12	0	24.50	0
Δv^1	0	0	0	0
Δv^1	-278.81	0	-272.27	0
Δz^1	4429.55	21741.57	0	0
Δz^2	0	0	0	0
Δh	15.07	-18.61	41.95	0

In *Table 3*, the negative values mean that the corresponding inputs or outputs must be decreased, equally. Also, the positive values mean that the corresponding inputs or outputs must be increased equally. For instance, the value of changing in undesirable output of stage 1 in unit 3 is $\Delta w=-91.12$ and it means that w must be reduced to 91.12. Moreover, x^2 must be increased by 364.49, because the value of changing in this input is positive.

In the first two cases, i.e. output changing of units 3 and 8, we observed that after increasing or decreasing outputs of the mentioned units, the efficiency value of the DMUs remained unchanged, while in the last case, i.e. output changing of units 5 and 10, due to dramatic increase of outputs of unit 8, the relative efficiency of some units are changed. For instance, unit 1 was efficient but after changing, it became inefficient. It should be noted that we cannot increase and decrease outputs (or inputs) arbitrarily. In fact, increasing or decreasing of outputs (or inputs) should be such that all outputs (or inputs) of the DMUs are less (or more) than or equal to all outputs (or inputs) of at least one non-dominated unit.

5| Conclusions

Recent studies on inverse DEA showed that this issue is an important subject that has attracted considerable attention among researchers. Inverse DEA allows the measuring the changes in a specific throughput (input or output) that occur due to the decrease or increase of other indicators while maintaining the level of efficiency. The importance of network structures with undesirable factors in various industries motivated us to study the inverse DEA model for a two-stage network structure with undesirable products.

In this regard, first, a two-stage network structure including undesirable factors is considered. Then, using the weak disposability assumption of Shephard [41], a model for evaluating the relative efficiency of the above-mentioned two-stage production system along with the efficiencies of each of its subsystems. In the following, an inverse DEA model for such systems is proposed. In the proposed model, the final output of the second stage is changed by a specific value and the amount of inputs changes and undesirable outputs are calculated while the level of efficiency of DMUs remains unchanged. It is noteworthy that the final output change can include an increase and a decrease simultaneously, and this is one of the strengths of the proposed method. After calculating the amount of changes in indicators for the second stage, the amount of changes in inputs and undesirable outputs for the first stage is calculated. Finally, the proposed approach has been illustrated by a real application on some poultry farms.

References

[1] Zhang, J., & Liu, Z. (1999). A further study on inverse linear programming problems. *Journal of Computational and Applied Mathematics*, *106*(2), 345-359. https://doi.org/10.1016/S0377-427(99)00080-1

[2] Wei, Q., Zhang, J., & Zhang, X. (2000). An inverse DEA model for inputs/outputs stimate. *European Journal of Operational Research*, *121*(1), 151-163. https://doi.org/10.1016/S0377-2217(99)00007-7

[3] Yan, H., Wei, Q., & Hao, G. (2002). DEA models for resource reallocation and production input/output estimation. *European Journal of Operational Research*, *136(1)*, 19-31. https://doi.org/10.1016/S0377-2217(01)00046-7

[4] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004). The outputs estimation of a DMU according to improvement of its efficiency. *Applied Mathematics and Computation*, *147(2)*, 409-413. https://doi.org/10.1016/S0096-3003(02)00734-8

[5] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004). Input estimation and identification of extra inputs in inverse DEA models. *Applied Mathematics and Computation*, *156(2)*, 427-437. https://doi.org/10.1016/j.amc.2003.08.001

[6] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2005). Sensitivity of efficiency classifications in the inverse DEA models. *Applied Mathematics and computation*, *169(2)*, 905-916. https://doi.org/10.1016/j.amc.2004.09.093

[7] Hadi-Vencheh, A., & Foroughi, A. A. (2006). A generalized DEA model for inputs/outputs estimation. *Mathematical and Computer Modelling*, *43(5-6)*, 447-457. https://doi.org/10.1016/j.amc.2004.09.093

[8] Alinezhad, A., Makui, A., & Mavi, R. K. (2007). An inverse DEA model for inputs/outputs estimation with respect to decision maker's preferences: The case of Refah bank of IRAN. *Mathematical Sciences, 1(1-2)*, 61-70.

[9] Lertworasirikul, S., Charnsethikul, P., & Fang, S. C. (2011). Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale. *Computers & Industrial Engineering, 61(4)*, 1017-1023. https://doi.org/10.1016/j.cie.2011.06.014

[10] Li, X., & Cui, J. (2008). A comprehensive DEA approach for the resource allocation problem based on scale economies classification. *Journal of Systems Science and Complexity, 21(4)*, 540.

[11] Li, X., & Cui, J. (2013). Inverse DEA model with considering returns to scale and elasticity. *In 11th International Symposium on Operations Research and its Applications in Engineering, Technology and Management 2013 (ISORA 2013)* (pp. 1-5). IET.

[12] Ghobadi, S., & Jahangiri, S. (2015). Inverse DEA: review, extension and application. *International Journal of Information Technology & Decision Making, 14(04)*, 805-824. https://doi.org/10.1142/S0219622014500370

[13] Gattoufi, S., Amin, G. R., & Emrouznejad, A. (2014). A new inverse DEA method for merging banks. *IMA Journal of Management Mathematics, 25(1)*, 73-87.

[14] Jahanshahloo, G. R., Soleimani-Damaneh, M., & Ghobadi, S. (2015). Inverse DEA under inter-temporal dependence using multiple-objective programming. *European Journal of Operational Research*, 240(2), 447-456. https://doi.org/10.1016/j.ejor.2014.07.002

[15] Eyni, M., Tohidi, G., & Mehrabeian, S. (2017). Applying inverse DEA and cone constraint to sensitivity analysis of DMUs with undesirable inputs and outputs. *Journal of the Operational Research Society, 68*, 34-40. https://doi.org/10.1057/s41274-016-0004-7

[16] Amin, G. R., Emrouznejad, A., & Gattoufi, S. (2017). Minor and major consolidations in inverse DEA: definition and determination. *Computers & Industrial Engineering, 103*, 193-200. https://doi.org/10.1016/j.cie.2016.11.029

[17] Amin, G. R., Emrouznejad, A., & Gattoufi, S. (2017). Modelling generalized firms' restructuring using inverse DEA. *Journal of Productivity Analysis, 48*, 51-61. https://doi.org/10.1007/s11123-017-0501-y

[18] Emrouznejad, A., Yang, G. L., & Amin, G. R. (2019). A novel inverse DEA model with application to allocate the CO2 emissions quota to different regions in Chinese manufacturing industries. *Journal of the Operational Research Society*, 70(7), 1079-1090. https://doi.org/10.1080/01605682.2018.1489344

[19] Ghiyasi, M., & Zhu, N. (2020). An inverse semi-oriented radial data envelopment analysis measure for dealing with negative data. *IMA Journal of Management Mathematics, 31(4)*, 505-516. https://doi.org/10.1093/imaman/dpaa007

[20] Lim, D. J. (2020). Inverse data envelopment analysis for operational planning: The impact of oil price shocks on the production frontier. *Expert Systems with Applications, 161*, 113726. https://doi.org/10.1016/j.eswa.2020.113726

[21] Färe, R., Grosskopf, S., & Lovell, C. (1985). The Measurement of Efficiency of Production. *Kluwer Nijhoff Publishing.*

[22] Seiford, L. M., & Zhu, J. (1999). Profitability and marketability of the top 55 US commercial banks. *Management science, 45(9)*, 1270-1288. https://doi.org/10.1287/mnsc.45.9.1270

[23] Kao, C., & Hwang, S. N. (2008). Efficiency decomposition in two-stage data envelopment analysis: An application to non-life insurance companies in Taiwan. *European journal of operational research, 185(1)*, 418-429. https://doi.org/10.1016/j.ejor.2006.11.041

[24] Liang, L., Cook, W. D., & Zhu, J. (2008). DEA models for two-stage processes: Game approach and efficiency decomposition. *Naval research logistics (NRL), 55(7)*, 643-653. https://doi.org/10.1002/nav.20308

[25] Chen, Y., Cook, W. D., Li, N., & Zhu, J. (2009). Additive efficiency decomposition in two-stage DEA. *European journal of operational research, 196(3)*, 1170-1176. https://doi.org/10.1016/j.ejor.2008.05.011

[26] Tone, K., & Tsutsui, M. (2009). Network DEA: A slacks-based measure approach. *European journal of operational research, 197(1)*, 243-252.

[27] Zha, Y., & Liang, L. (2010). Two-stage cooperation model with input freely distributed among the stages. *European Journal of Operational Research, 205(2)*, 332-338. https://doi.org/10.1016/j.ejor.2010.01.010

[28] Chen, Y., Du, J., Sherman, H. D., & Zhu, J. (2010). DEA model with shared resources and efficiency decomposition. *European Journal of Operational Research, 207(1)*, 339-349. https://doi.org/10.1016/j.ejor.2010.03.031

[29] Amirteimoori, A., Kordrostami, S., & Azizi, H. (2016). Additive models for network data envelopment analysis in the presence of shared resources. *Transportation Research Part D: Transport and Environment*, *48*, 411-424. https://doi.org/10.1016/j.ejor.2010.03.031

[30] Fukuyama, H., & Weber, W. L. (2010). A slacks-based inefficiency measure for a two-stage system with bad outputs. *Omega*, *38(5)*, 398-409. https://doi.org/10.1016/j.omega.2009.10.006

[31] Lozano, S., Gutiérrez, E., & Moreno, P. (2013). Network DEA approach to airports performance assessment considering undesirable outputs. *Applied Mathematical Modelling*, *37(4)*, 1665-1676. https://doi.org/10.1016/j.apm.2012.04.041

[32] Maghbouli, M., Amirteimoori, A., & Kordrostami, S. (2014). Two-stage network structures with undesirable outputs: A DEA based approach. *Measurement*, *48*, 109-118. https://doi.org/10.1016/j.measurement.2013.10.032

[33] Amirteimoori, A., Toloie-Eshlaghi, A., & Homayoonfar, M. (2014). Efficiency measurement in Two-Stage network structures considering undesirable outputs. *International Journal of Industrial Mathematics*, *6(1)*, 65-71.

[34] Wu, J., Zhu, Q., Ji, X., Chu, J., & Liang, L. (2016). Two-stage network processes with shared resources and resources recovered from undesirable outputs. *European Journal of Operational Research*, *251(1)*, 182-197. https://doi.org/10.1016/j.ejor.2015.10.049

[35] Nematizadeh, M., & Nematizadeh, M. (2021). Profitability assessment of two-stage feedback processes with undesirable factors. *Journal of applied research on industrial engineering*, *8(Special Issue)*, 1-16. https://doi.org/10.22105/jarie.2021.278230.1278

[36] Kalantary, M., Farzipoor Saen, R., & Toloie Eshlaghy, A. (2018). Sustainability assessment of supply chains by inverse network dynamic data envelopment analysis. *Scientia Iranica*, *25(6)*, 3723-3743.

[37] Kalantary, M., & Saen, R. F. (2019). Assessing sustainability of supply chains: An inverse network dynamic DEA model. *Computers & Industrial Engineering*, *135*, 1224-1238. https://doi.org/10.1016/j.cie.2018.11.009

[38] Amin, G. R., & Ibn Boamah, M. (2021). A two-stage inverse data envelopment analysis approach for estimating potential merger gains in the US banking sector. *Managerial and Decision Economics*, *42(6)*, 1454-1465. https://doi.org/10.1002/mde.3319

[39] Moghaddas, Z., Tosarkani, B. M., & Yousefi, S. (2022). Resource reallocation for improving sustainable supply chain performance: An inverse data envelopment analysis. *International Journal of Production Economics*, *252*, 108560. https://doi.org/10.1016/j.ijpe.2022.108560

[40] Arbabi, M., Moghaddas, Z., Amirteimoori, A., & Khunsiavash, M. (2022). An innovative inverse model of network data envelopment analysis. *Journal of Applied Research on Industrial Engineering*. https://doi.org/10.22105/jarie.2022.326607.1445

[41] Shephard, R. W. (2015). Theory of cost and production functions. Princeton University Press.

[42] Yu, G., Wei, Q., & Brockett, P., (1996). A generalized data envelopment analysis model: A unification and extension of existing methods.

[43] Yu, G., Wei, Q., (1997). Analyzing properties of K-cones in the generalized data envelopment analysis model. *Journal of Econometrics*, *80(1)*, 63-84. https://doi.org/10.1016/S0304-4076(97)00003-1

[44] Färe, R., & Grosskopf, S. (2003). Nonparametric productivity analysis with undesirable outputs: Comment. *American Journal of Agricultural Economics*, *85(4)*, 1070-1074.

[45] Kuosmanen, T. (2005). Weak disposability in nonparametric production analysis with undesirable outputs. *American Journal of Agricultural Economics*, *87(4)*, 1077-1082. https://doi.org/10.1111/j.1467-8276.2005.00788.x