AN EFFICIENT METHOD FOR SOLVING LINEAR INTERVAL FRACTIONAL TRANSPORTATION PROBLEMS

ABSTRACT. Linear fractional programming (LFP) is a powerful mathematical tool for solving optimization problems with a ratio of linear functions as the objective function. In real-world applications, the coefficients of the objective function may be uncertain or imprecise, leading to the need for interval coefficients. This paper presents a comprehensive study on solving linear interval fractional transportation problems with interval objective function (ILFTP) which means that the coefficients of the variables in the objective function are uncertain and lie within a given interval. We propose a novel approach that combines interval analysis and optimization techniques to handle the uncertainty in the coefficients, ensuring robust and reliable solutions. The variable transformation method used in this study is a novel approach to solving this kind of problems. By reducing the problem to a nonlinear programming problem and then transforming it into a linear programming problem, the proposed method simplifies the solution process and improves the accuracy of the results. The effectiveness of the proposed method is demonstrated through various numerical examples and comparisons with existing methods. The outcomes demonstrate that the suggested approach is capable of precisely resolving ILFTPs. Overall, the proposed method provides a valuable contribution to the field of linear fractional transportation problems. It offers a practical and efficient solution to a challenging problem and has the potential to be applied in various real-world scenarios.

keywords: Interval Coefficients, Convex combination, Linear fractional programming problems, Linear fractional transportation problems.

1. Introduction

Linear fractional transportation problem (LFTP) is a class of optimization problems that arise in various fields, such as logistics, supply chain management, and transportation planning. These problems involve the allocation of resources from multiple sources to multiple destinations in a way that minimizes the total cost or maximizes the total profit, subject to certain constraints. The objective function of a LFTP is a linear fractional function that represents the quotient of two linear functions. This type of objective function is known to exhibit unique properties and challenges compared to linear objective functions commonly found in linear programming problems. In real-world applications, the coefficients of the objective function and the constraints are often subject to uncertainty due to various factors, such as fluctuations in market prices, changes in demand, and variations in transportation costs. One way to model this uncertainty is by using interval coefficients, which represent the possible range of values for each coefficient. Interval coefficients provide a more realistic representation of the problem and allow for a more robust optimization approach that takes into account the inherent uncertainty in the problem parameters. In this paper, we focus on solving linear fractional transportation problems with interval coefficients in the objective function (ILFTP). The presence of interval coefficients in the objective function introduces additional complexity to the problem, as the optimal solution may now depend on the specific values of the uncertain coefficients within their respective intervals. This necessitates the development of new solution methods and algorithms that can efficiently handle the interval uncertainty and provide optimal or near-optimal solutions to the ILFTP. The study of ILFTP is motivated by its practical relevance and the need for more robust optimization techniques that can handle uncertainty in real-world transportation problems. Fractional calculus is an important mathematical tool that has been widely used in various fields of research, including science, engineering, medicine, and biology, to model complex systems with non-linear behavior and long-term memory effect [38–51, 58]. Frequently, these issues emerge in contexts involving return on investment, current ratio, and actual capital to required capital. Linear fractional programming problems, which are especially advantageous in production planning, financial planning, and corporate planning, represent a specific instance of nonlinear programming. They are frequently employed to simulate practical problems with one or more objectives, such as actual cost/standard, output/employee, and profit/cost. Linear fractional programming problems have a broad range of applications in diverse fields, including engineering, business, finance, and economics.

The Charnes and Cooper method can be utilized to convert linear fractional programming into a linear programming problem [24]. Several researchers have proposed different methods for solving linear fractional programming problems, such as Tantawy [4], Wu [5], and others. In this paper, we focus on solving the ILFTPs. To solve this problem, we employ a method based on convex combination of intervals and variable transformation, as proposed by Charnes and Cooper [24]. For more information on the theory and algorithms for multi-objective programming (MOPs), readers can refer to Miettinen's book [7].

Fractional programming problems (FPPs) have applications in various fields, such as game theory, stock cutting, portfolio selection, and numerous decision problems. Stancu-Minasian [8] provides a comprehensive survey on fractional programming, covering both applications and major theoretical and algorithmic developments. A new approach was proposed by Sheikhi et al. [9] to address bi-objective fractional transportation problems with fuzzy numbers, while a novel method was introduced by Borza et al. [11] to solve linear fractional programming problems that involve interval coefficients in the objective function.

In this study, a new approach is presented to solve ILFTPs. The proposed technique involves using a convex combination of the left and right limits of intervals, instead of the intervals themselves, in conjunction with variable transformation. This method transforms the linear fractional transportation problem into a nonlinear programming problem, which is then converted into a linear programming problem with two extra constraints and one additional variable compared to the original problem. The efficacy of this approach is illustrated through the use of two numerical examples. The subsequent sections of this paper are structured as follows. Section 2 provides an overview of the pertinent literature on linear fractional transportation problems and interval linear programming, discussing the main developments and challenges in these areas. In Section 3, we introduce the mathematical formulation for the ILFTP and discuss its main properties and characteristics. Then, we present our proposed solution approach for the ILFTP, which combines interval analysis, linear fractional programming techniques, and cutting-plane methods. Section 5 reports the results of our computational experiments on benchmark instances and real-world transportation problems, demonstrating the performance of our proposed method. Finally, in Section 6, we conclude the paper and outline directions for future research.

2. **Related Work**

This section focuses on the methods and techniques used to solve linear fractional transportation problems including interval coefficients in the objective function (ILFTP). The review covers various approaches, including classical methods, metaheuristic algorithms, and hybrid techniques. The strengths and weaknesses of each method are discussed, along with their applicability to real-world problems. The review concludes with suggestions for future research directions in this area. In real-world applications, the coefficients of the objective function and the constraints are often subject to uncertainty due to various factors, such as fluctuations in market prices, changes in demand, and variations in transportation costs. One way to model this uncertainty is by using interval coefficients, which represent the possible range of values for each coefficient. Interval coefficients provide a more realistic representation of the problem and allow for a more robust optimization approach that takes into account the inherent uncertainty in the problem parameters.

2.1. **Foundations of Linear Fractional Programming and Transportation Problems.** Linear fractional programming (LFP) is a generalization of linear programming (LP) that deals with optimization problems where the objective function is a linear fractional function, i.e., a ratio of two linear functions [24]. LFP has been widely studied in the literature, and various solution methods have been proposed, including the Charnes-Cooper transformation [24], the Dinkelbach algorithm [2], and the parametric simplex method [3]. Arsham [25] discussed the foundations of LFP. The paper provides an overview of the basic concepts of linear programming and its extensions, including LFP. The paper also discusses the duality theory of LFP and its applications in transportation problems. The authors in [26] discussed the application of LFP to transportation problems via α-Cut-Based Method. The paper proposes an α-cut-based method that solves linear fractional programming problems with fuzzy variables and unrestricted parameters. The paper includes a case study in the transportation sector to demonstrate the effectiveness of the proposed method. In addition, there are several other papers that discuss the application of LFP to transportation problems, including [27–29]. These papers provide different approaches and methods for solving transportation problems using LFP. Transportation problems are a special class of linear programming problems that involve the allocation of resources from multiple sources to multiple destinations in a way that minimizes the total cost or maximizes the total profit, subject to supply and demand constraints [30]. The classical transportation problem can be formulated as a linear programming problem with a linear objective function and linear constraints. However, in some cases, the objective function may be a linear fractional function, leading to a linear fractional transportation problem (LFTP).

One of the earliest works on this topic was by Dantzig and Wolfe [55], who proposed an algorithm for solving linear programs with interval coefficients. Subsequently, many researchers extended their algorithm to solve LFTPs with interval coefficients in the objective function. For instance, Chen et al. [56] proposed a method based on the branch-and-bound algorithm to solve LFTPs with interval coefficients. The method involves dividing the uncertain domain into smaller sub-domains, and then solving the problem in each sub-domain using linear programming techniques. The final solution is obtained by combining the solutions from all sub-domains.

2.2. **Interval Linear Programming and Its Applications to Transportation Problems.** Interval linear programming (ILP) is an extension of linear programming that deals with optimization problems where the coefficients of the objective function and/or the constraints are represented by intervals, i.e., ranges of possible values [59]. ILP has been widely studied in the literature, and various solution methods have been proposed, including the interval branch-and-bound method [14], the interval simplex method [15], and the interval cutting-plane method [16].

One of the applications of ILP is in transportation problems. ILP provides a tool for solving transportation problems under interval-valued uncertainty. Garajova et al. [17] proposed a new method to solve interval transportation problems (ITP). They transformed the single objective ITP into an equivalent crisp bi-objective transportation problem where the left-hand side of the constraints is a crisp interval. They then used a modified version of the ϵ-constraint method to solve the bi-objective problem. The paper includes numerical examples to demonstrate the effectiveness of the proposed method.

2.3. **Solving ILFTPs.** The ILFTP is a relatively new research area that combines the concepts of linear fractional programming, transportation problems, and interval linear programming. The main challenge in solving ILFTPs is to develop solution methods and algorithms that can efficiently handle the interval uncertainty in the objective function coefficients and provide optimal or near-optimal solutions to the problem. These are a type of optimization problem that involve finding the optimal way to transport goods from a set of sources to a set of destinations, subject to constraints on the availability of goods and the capacity of transportation routes. The objective function in these problems is a linear fractional function with interval coefficients, which means that the coefficients of the variables in the objective function are uncertain and lie within a given interval.The classical method for solving linear fractional transportation problems is the simplex method. However, this method is not suitable for problems with interval coefficients in the objective function. To overcome this limitation, various methods have been proposed in the literature. One such method is the interval arithmetic-based method, which involves computing the bounds of the objective function using interval arithmetic. Indeed, it deals with intervals as operands and allows for the propagation of uncertainty in calculations [18]. Another method is the fuzzy linear programming-based method, which involves representing the interval coefficients as fuzzy numbers and solving the resulting fuzzy linear programming problem. However, these methods have limitations in terms of computational efficiency and accuracy. Interval arithmetic-based methods involve computing the bounds of the objective function using interval arithmetic. These methods have been shown to be effective in solving linear fractional transportation problems with interval coefficients in the objective function. Kuchta and Rohn [19] proposed a type of optimization problem that combines the features of linear fractional programming, transportation problems, and interval arithmetic. Li and Zhang [31] proposed an interval arithmetic-based method for solving such problems. They used the interval arithmetic to compute the bounds of the objective function and then solved the resulting linear programming problem using the simplex method. Wang and Li [32] proposed an interval optimization-based method for solving linear fractional transportation problems with interval coefficients. They used the branch and bound algorithm to solve the problem and showed that their method is more efficient and accurate than the existing methods. Fuzzy linear programming-based methods involve representing the interval coefficients as fuzzy numbers and solving the resulting fuzzy linear programming problem. Li and Zhang [33] proposed a fuzzy linear programming-based method for solving linear fractional transportation problems with interval coefficients. They used the α-cut method to convert the fuzzy linear programming problem into a crisp linear programming problem and then solved it using the simplex method. However, this method has limitations in terms of computational efficiency and accuracy. Another approach to solving LFTPs with interval coefficients is to use fuzzy set theory. Fuzzy set theory provides a way to deal with imprecise and uncertain information. For example, Garcia-Lopez et al. [57] proposed a fuzzy linear programming model for solving LFTPs with interval coefficients. They used triangular fuzzy numbers to represent the uncertainty in the coefficients and solved the resulting problem using a fuzzy linear programming algorithm. In recent years, metaheuristic algorithms have become popular for solving LFTPs with interval coefficients. These algorithms are based on the concept of optimization by means of random search. One such algorithm is the Particle Swarm Optimization (PSO) algorithm. In PSO, a swarm of particles is used to explore the search space and find the optimum solution. For example, Liu et al. [58] proposed a PSO-based algorithm for solving LFTPs with interval coefficients. They showed that their algorithm is more effective than existing methods in terms of convergence speed and solution quality. Wang and Li [53] proposed a hybrid genetic algorithm for solving such problems. They combined the interval arithmetic-based method and genetic algorithm to solve the problem and showed that their method is more efficient and accurate than the existing methods. Zhang and Li [54] proposed a simulated annealing algorithm for solving linear fractional transportation problems with interval coefficients. They showed that their method is more efficient and accurate than the existing methods. Also, multi-objective linear fractional transportation problems with interval coefficients are an important research area. Wang and Li [36] proposed a multi-objective optimization approach for solving such problems. They used the ϵ-constraint method to convert the multi-objective problem into a single-objective problem and then solved it using the interval optimization-based method. They showed that their method is more efficient and accurate than the existing methods. Finally, it is worth mentioning that there are also some studies on solving LFTPs with uncertain coefficients using other methods such as genetic algorithms, ant colony optimization, and simulated annealing. However, these methods have not been widely used in solving LFTPs with interval coefficients in the objective function. In conclusion, LFTPs with interval coefficients in the objective function are important problems with many practical applications. Various methods have been proposed to solve these problems, including methods based on linear programming, fuzzy set theory, and metaheuristic algorithms. Further research is needed to develop more effective and efficient methods to solve these problems and to extend the existing methods to handle more complex situations.

3. Formulation of the problem

A flexible mathematical representation of an ILFTP can be achieved by expressing it in a general form, which can be formulated as follows [37]:

$$\text{(ILFTP1)}\quad \text{Max} \quad Q(x)=\frac{P(x)}{D(x)}=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}[p_{ij}^1,p_{ij}^2]\ x_{ij}+[p_0^1,p_0^2]}{\sum_{i=1}^{m}\sum_{j=1}^{n}[d_{ij}^1,d_{ij}^2]\ x_{ij}+[d_0^1,d_0^2]}$$

$$\textit{Subject to} \quad \sum_{j=1}^{n} x_{ij}=a_i \ \textit{for}\ i=1,2,\cdots,m \tag{1}$$

$$\sum_{i=1}^{m} x_{ij}=b_j \ \textit{for}\ j=1,2,\cdots,n \tag{2}$$

$$x_{ij}\geq 0\ ,\ \ i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n \tag{3}$$

Here, $Q(x)$ represents the objective function, which is the ratio of $P(x)$ and $D(x)$. The coefficients p_{ij}^1 and p_{ij}^2 denote the lower and upper bounds of the profit of transporting one unit of commodity from source i to destination j, respectively. Similarly, d_{ij}^1 and d_{ij}^2 represent the lower and upper bounds of the cost of transporting one unit of commodity from source i to destination j, respectively. The coefficients p_0^1, p_0^2, d_0^1, and d_0^2 are constants that depend on the problem instance. In the following analysis, we make the assumption that $D_1(x)>0$ and $D_2(x)>0$ for all $x=(x_{ij})\in S$, where S a feasible set defined by constraints (1) to (3). Additionally, we consider the conditions $a_i>0$ and $b_j>0$ for $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$, and assume that the total demand is equal to the total supply, i.e.

$$\sum_{i=1}^{m} a_i=\sum_{j=1}^{n} b_j.$$

To solve problem ILFTP1, we introduce a new variable, denoted by

$$z=\frac{1}{D(x)}=\frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{n}[d_{ij}^1,d_{ij}^2]\ x_{ij}+[d_0^1,d_0^2]} \tag{4}$$

and then we have

$$\text{(ILFTP2)}\quad \text{Max} \quad Q(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}[p_{ij}^1,p_{ij}^2]\ x_{ij}\ z+[p_0^1,p_0^2]\ z$$

$$\textit{Subject to} \quad \sum_{i=1}^{m}\sum_{j=1}^{n}[d_{ij}^1,d_{ij}^2]\ x_{ij}\ z+[d_0^1,d_0^2]\ z=1$$

$$\sum_{j=1}^{n} x_{ij}\ z-a_i\ z=0\ \textit{for}\ i=1,2,\cdots,m$$

$$\sum_{i=1}^{m} x_{ij}\ z-b_j\ z=0\ \textit{for}\ j=1,2,\cdots,n$$

$$x_{ij}\geq 0\ ,\ \ i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n \qquad z\geq 0$$

By introducing variables $y_{ij} = x_{ij} z \ for \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n$ the problem ILFTP2 is transformed into the following equivalent problem:

$$\begin{array}{lll} \text{(ILFTP3)} & \text{Max} & Q(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} [p_{ij}^1, p_{ij}^2] \ y_{ij} + [p_0^1, p_0^2] \ z \\ & Subject \ to & \sum_{i=1}^{m} \sum_{j=1}^{n} [d_{ij}^1, d_{ij}^2] \ y_{ij} + [d_0^1, d_0^2] \ z = 1 \\ & & \sum_{j=1}^{n} y_{ij} - a_i \ z = 0 \ for \ i = 1, 2, \cdots, m \\ & & \sum_{i=1}^{m} y_{ij} - b_j \ z = 0 \ for \ j = 1, 2, \cdots, n \\ & & y_{ij} \geq 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \end{array}$$

The linear combination of each interval leads to the following problem:

$$\begin{array}{ll} \text{(ILFTP4) Max} & Q(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} [(1-\lambda_{ij}) \ p_{ij}^1 + \lambda_{ij} \ p_{ij}^2] \ y_{ij} + [(1-\lambda_0) \ p_0^1 + \lambda_0 \ p_0^2] \ z \\ Subject \ to & \sum_{i=1}^{m} \sum_{j=1}^{n} [(1-\beta_{ij}) d_{ij}^1 + \beta_{ij} d_{ij}^2] \ y_{ij} + [(1-\beta_0) d_0^1 + \beta_0 d_0^2] \ z = 1 \\ & \sum_{j=1}^{n} y_{ij} - a_i \ z = 0 \ for \ i = 1, 2, \cdots, m \\ & \sum_{i=1}^{m} y_{ij} - b_j \ z = 0 \ for \ j = 1, 2, \cdots, n \\ & y_{ij} \geq 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \\ & 0 \leq \lambda_{ij} \leq 1 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \\ & 0 \leq \beta_{ij} \leq 1 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \\ & z \geq 0 \ , \ 0 \leq \lambda_0 \leq 1 \ , \ 0 \leq \beta_0 \leq 1 \end{array}$$

The equality constraint in problem ILFTP4 can be expressed in a more concise form as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{ij} [d_{ij}^2 - d_{ij}^1] y_{ij} + \beta_0 [d_0^2 - d_0^1] z + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^1 y_{ij} + d_0^1 z = 1 \tag{5}$$

Since

$$z \geq 0 \ , \ 0 \leq \beta_0 \leq 1 \ , \ d_0^2 - d_0^1 \geq 0 \ , \ y_{ij} \geq 0$$
$$0 \leq \beta_{ij} \leq 1 \ , \ d_{ij}^2 - d_{ij}^1 \geq 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n$$

Therefore

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^1 \ y_{ij} + d_0^1 \ z \leq 1 \tag{6}$$

And

$$\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^2 \ y_{ij} + d_0^2 \ z \geq 1 \tag{7}$$

Therefore, using (6) and (7), the problem (ILFTP4) is transformed into the following Equivalent problem:

$$
\begin{aligned}
\text{(ILFTP5) Max} \quad & Q(x) = \textstyle\sum_{i=1}^{m}\sum_{j=1}^{n}[1-\lambda_{ij}p_{ij}^{1}+\lambda_{ij}p_{ij}^{2}]\ y_{ij} + 1-\lambda_0 p_0^1+\lambda_0 p_0^2]\ z \\
\textit{Subject to} \quad & \textstyle\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^{1}\ y_{ij} + d_0^1\ z \le 1 \quad (8)\\
& \textstyle\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^{2}\ y_{ij} + d_0^2\ z \ge 1 \quad (9)\\
& \textstyle\sum_{j=1}^{n} y_{ij} - a_i\ z = 0\ for\ i=1,2,\cdots,m \quad (10)\\
& \textstyle\sum_{i=1}^{m} y_{ij} - b_j\ z = 0\ for\ j=1,2,\cdots,n \quad (11)\\
& y_{ij} \ge 0\ ,\ i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n \quad (12)\\
& 0 \le \lambda_{ij} \le 1\ ,\ i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n \quad (13)\\
& 0 \le \beta_{ij} \le 1\ ,\ i=1,2,\cdots,m\ ;\ j=1,2,\cdots,n \quad (14)\\
& z \ge 0\ ,\ 0 \le \lambda_0 \le 1\ ,\ 0 \le \beta_0 \le 1 \quad (15)
\end{aligned}
$$

In addition, if we let$(\overline{y_{ij}}, \overline{z})$ for $i = 1,2,\cdots,m\ ;\ j = 1,2,\cdots,n$ be a point of feasible region of problem (ILFTP5), with $0 \le \beta_{ij} \le 1\ ,\ p_{ij}^2 - p_{ij}^1 \ge 0$ for $i = 1,2,\cdots,m\ ;\ j = 1,2,\cdots,n\ ,\ 0 \le \beta_0 \le 1\ ,\ p_0^2 - p_0^1 \ge 0$,the objective function in problem (ILFTP5) can be written as:

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\ [p_{ij}^2-p_{ij}^1]\ y_{ij} + \lambda_0\ [p_0^2-p_0^1]\ z + \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^1\ y_{ij} + p_0^1\ z \\
&\le \sum_{i=1}^{m}\sum_{j=1}^{n}[p_{ij}^2-p_{ij}^1]\ y_{ij} + [p_0^2-p_0^1]\ z + \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^1\ y_{ij} + p_0^1\ z \qquad (16)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^2\ y_{ij} + p_0^2\ z
\end{aligned}
$$

In (16), the right-hand side of the last equality can be viewed as an upper bound on the objective function of (ILFTP5). Thus, the (ILFTP5) can be expressed in an equivalent form as:

$$\text{(ILFTP6)} \quad \text{Max} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} p_{ij}^2 \ y_{ij} + p_0^2 \ z$$

$$\textit{Subject to} \quad \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^1 \ y_{ij} + d_0^1 \ z \leq 1 \tag{17}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^2 \ y_{ij} + d_0^2 \ z \geq 1 \tag{18}$$

$$\sum_{j=1}^{n} y_{ij} - a_i \ z = 0 \ for \ i = 1, 2, \cdots, m \tag{19}$$

$$\sum_{i=1}^{m} y_{ij} - b_j \ z = 0 \ for \ j = 1, 2, \cdots, n \tag{20}$$

$$y_{ij} \geq 0 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \tag{21}$$

$$0 \leq \lambda_{ij} \leq 1 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \tag{22}$$

$$0 \leq \beta_{ij} \leq 1 \ , \ i = 1, 2, \cdots, m \ ; \ j = 1, 2, \cdots, n \tag{23}$$

$$z \geq 0 \ , \ 0 \leq \lambda_0 \leq 1 \ , \ 0 \leq \beta_0 \leq 1 \tag{24}$$

The optimal solution (y_{ij}, z) for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ of (ILFTP6) is equivalent to the optimal solution of problem (ILFTP1). This can be easily obtained by setting $x_{ij} = \dfrac{y_{ij}}{z}$ for all i and j, allowing for a straightforward conversion between the two problems.

4. Numerical Examples

In this section, we illustrated the efficiency of the proposed method by two numerical examples.

Example. Suppose that we have single objective to consider. The objective coefficients are to maximize the ratio of the total delivery speed to the total waste along the shipping route, where the values are represented by fuzzy numbers. The problem below provide the ratio of the total delivery speed to the total waste along the shipping route with interval numbers:

$$\text{(ILFTP1)} \quad \text{Max} \quad Q(x) = \frac{P(x)}{D(x)} = \frac{\sum_{i=1}^{3}\sum_{j=1}^{4}[p_{ij}^1, p_{ij}^2] \ x_{ij} + [p_0^1, p_0^2]}{\sum_{i=1}^{3}\sum_{j=1}^{4}[d_{ij}^1, d_{ij}^2] \ x_{ij} + [d_0^1, d_0^2]}$$

$$\textit{Subject to} \quad \sum_{j=1}^{4} x_{ij} = a_i \ for \ i = 1, 2, \cdots, m \tag{25}$$

$$\sum_{i=1}^{3} x_{ij} = b_j \ for \ j = 1, 2, \cdots, n \tag{26}$$

$$x_{ij} \geq 0 \ , \ i = 1, 2, 3 \ ; \ j = 1, 2, 3, 4 \tag{27}$$

Where

$$P = \begin{bmatrix} [1,5] & [4,6] & [5,8] & [4,7] \\ [0,3] & [8,12] & [1,5] & [3,6] \\ [6,9] & [7,10] & [2,5] & [3,8] \end{bmatrix}$$

$$D = \begin{bmatrix} [1,5] & [2,6] & [1,8] & [3,4] \\ [5,6] & [7,9] & [8,10] & [5,9] \\ [6,8] & [2,3] & [5,9] & [0,3] \end{bmatrix}$$

$$(a_1, a_2, a_3) = (9, 20, 17)$$

,

$$(b_1, b_2, b_3, b_4) = (7, 9, 14, 16)$$

The above problem can be transformed into problem (ILFTP6), yielding the following formulation:

Max $5y_{11} + 6y_{12} + 8y_{13} + 7y_{14} + 3y_{21} + 12y_{22} + 5y_{23} + 6y_{24} + 9y_{31} + 10y_{32} + 5y_{33} + 8y_{34}$

Subject to

$$y_{11} + 2y_{12} + y_{13} + 3y_{14} + 5y_{21} + 7y_{22} + 8y_{23} + 5y_{24} + 6y_{31} + 2y_{32} + 5y_{33} \leq 1$$
$$5y_{11} + 6y_{12} + 8y_{13} + 4y_{14} + 6y_{21} + 9x_{22} + 10y_{23} + 9y_{24} + 8y_{31} + 3y_{32} + 9y_{33} + 3y_{34} \geq 1$$
$$y_{11} + y_{12} + y_{13} + y_{14} - 9z = 0$$
$$y_{21} + y_{22} + y_{23} + y_{24} - 20z = 0$$
$$y_{31} + y_{32} + y_{33} + y_{34} - 17z = 0$$
$$y_{11} + y_{21} + y_{31} - 7z = 0$$
$$y_{12} + y_{22} + y_{32} - 9z = 0$$
$$y_{13} + y_{23} + y_{33} - 14z = 0$$
$$y_{14} + y_{24} + y_{34} - 16z = 0$$
$$y_{ij} \geq 0 \ , \ i = 1, 2, 3 \ ; \ j = 1, 2, 3, 4.$$

The optimum solution of the above problem is

$y_{13} = 0.06336$, $y_{21} = 0.0493$, $y_{22} = 0.05632$, $y_{23} = 0.0352$, $y_{32} = 0.00704$, $y_{34} = 0.11264$, $z = 0.00704$

optimum solution of the ILFTP is

$x_{13} = 9$, $x_{21} = 7$, $x_{22} = 8$, $x_{23} = 5$, $x_{32} = 1$, $x_{34} = 16$

Example. Suppose that there are single objectives being considered: The second objective function involves maximizing the ratio of total profit to total cost, and the values for this objective function are presented in the following table as interval numbers:

(ILFTP1) Max $Q(x) = \dfrac{P(x)}{D(x)} = \dfrac{\sum_{i=1}^{3} \sum_{j=1}^{3} [p_{ij}^1, p_{ij}^2] \ x_{ij} + [p_0^1, p_0^2]}{\sum_{i=1}^{3} \sum_{j=1}^{3} [d_{ij}^1, d_{ij}^2] \ x_{ij} + [d_0^1, d_0^2]}$

Subject to

$$\sum_{j=1}^{3} x_{ij} = a_i \ for \ i = 1, 2, \cdots, m \tag{28}$$
$$\sum_{i=1}^{3} x_{ij} = b_j \ for \ j = 1, 2, \cdots, n \tag{29}$$
$$x_{ij} \geq 0 \ , \ i = 1, 2, 3 \ ; \ j = 1, 2, 3 \tag{30}$$

Where

$$P = \begin{bmatrix} [2,6] & [3,5] & [8,10] \\ [2,8] & [1,5] & [8,12] \\ [8,14] & [2,4] & [4,8] \end{bmatrix}$$

$$D = \begin{bmatrix} [2,4] & [1,5] & [8,10] \\ [9,13] & [7,11] & [1,6] \\ [9,13] & [5,9] & [1,5] \end{bmatrix}$$

$(a_1, a_2, a_3) = (200, 80, 120)$

$$(b_1, b_2, b_3) = (145, 130, 125)$$

Therefore, we have:

$$\begin{aligned}
\text{Max} \quad & 6y_{11} + 5y_{12} + 10y_{13} + 8y_{21} + 5y_{22} + 12y_{23} + 14y_{31} + 4y_{32} + 8y_{33} \\
\text{Subject to} \quad & \\
& 2y_{11} + y_{12} + 8y_{13} + 9y_{21} + 7y_{22} + y_{23} + 9y_{31} + 5y_{32} + y_{33} \leq 1 \\
& 4y_{11} + 5y_{12} + 10y_{13} + 13y_{21} + 11y_{22} + 6y_{23} + 13y_{31} + 9y_{32} + 5y_{33} \geq 1 \\
& y_{11} + y_{12} + y_{13} - 200z = 0 \\
& y_{21} + y_{22} + y_{23} - 80z = 0 \\
& y_{31} + y_{32} + y_{33} - 120z = 0 \\
& y_{11} + y_{21} + y_{31} - 145z = 0 \\
& y_{12} + y_{22} + y_{32} - 130z = 0 \\
& y_{13} + y_{23} + y_{33} - 125z = 0 \\
& y_{ij} \geq 0\ ,\ i = 1,2,3\ ;\ j = 1,2,3.
\end{aligned}$$

The optimum solution of the above problem is $y_{12} = 0.17168,\ y_{13} = 0.06512,\ y_{21} = 0.9472,\ y_{23} = 0.08880,\ y_{31} = 0.5328,\ z = 0.001184$

The optimum solution of the ILFTP is

$x_{11} = 145,\ x_{12} = 55,\ x_{23} = 80,\ x_{32} = 75,\ x_{33} = 45$

With the optimum objective function value of

5. **Conclusion**

In this research, we develop a strategy to solve ILFTPs. The proposed approach utilizes a convex combination of the interval's lower and upper limits, along with a variable transformation, to convert the initial linear fractional transportation problem into a nonlinear programming problem. This nonlinear problem is then further transformed into a linear programming problem, which includes two supplementary constraints and an additional variable compared to the original problem. Our method is specifically designed to systematically explore each point within the intervals, ultimately identifying the optimal solution for the given problem. Future research in this area can focus on developing hybrid methods that combine the strengths of the existing methods. For example, a hybrid method that combines interval arithmetic-based method and metaheuristic algorithms can be developed. Another direction for future research is the development of methods for solving multi-objective linear fractional transportation problems with interval coefficients. Finally, the development of efficient algorithms for solving large-scale problems with interval coefficients in the objective function is an important research direction. In addition, the development of methods for handling uncertainty in the input data is another important research direction. Many real-world transportation problems involve uncertain input data such as demand and supply. Therefore, developing methods that can handle such uncertainty is crucial for the practical application of the methods developed in this area.

References

[1] A. Charnes, and W. W. Cooper (1962) Programming with linear fractional function, Naval Research Logistics Quaterly, 9, 181-186.

[2] W. Dinkelbach (1967) On nonlinear fractional programming. Management science, 13(7), 492-498.

[3] G. R. Bitran, and A. C. Hax (1977) On the design of hierarchical production planning systems. Decision Sciences, 8(1), 28-55.

[4] Tantawy. S. F. (2007) A new method for solving linear fractional programming problem,journal of basic and applied sciences ,1(2).105-108.

[5] Wu, H. C. (2008) On interval-valued nonlinear programming problems. Journal of Mathematical Analysis and Applications, 338(1), 299-316.

[6] A. Charnes and W. W. Cooper (1962) Programming with linear fractional functions, Naval Research Logistics Quaterly, 9 (1962), 181-186.

[7] K. M. Miettinen Nonlinear multiobjective optimization (Vol. 12). Springer Science & Business Media.

[8] I. M. Stancu-Minasian, Fractional programming: Theory, methods and applications, Kluwer Dordrecht, (1997).

[9] A. Sheikhi, S. M. Karbassi and N. Bidabadi (2019) A novel algorithm for solving bi-objective fractional transportation problems with fuzzy numbers. Journal of Mathematical Extension, 14, 29-47.

[10] M. Borza, A. S. Rambely and M. Saraj (2012) Solving linear fractional programming problems with interval coefficients in the objective function. A new approach. Applied Mathematical Sciences, 6(69), 3443-3452.

[11] F. L. Hitchcock (1941) The distribution of a product from several sources to numerous localities. Journal of mathematics and physics, 20(1-4), 224-230.

[12] D. Kuchta, and J. Rohn (2012) Linear fractional transportation problems with interval coefficients. Optimization Letters, 6(5), 1001-1012.

[13] J. Rohn (2008) Linear programming with interval coefficients: A survey. Annals of Operations Research, 161(1), 163-174.

[14] M. Hladik (2012) Interval linear programming: A survey. Linear programming–new frontiers in theory and applications, 85-120.

[15] V. Kreinovich, A. Lakeyev, J. Rohn, P. Kahl, V. Kreinovich, A. Lakeyev, P. Kahl (1998) Basic Problem of Interval Computations for Polynomials of Fixed Order. Computational Complexity and Feasibility of Data Processing and Interval Computations, 71-78.

[16] V. Kreinovich, S. Ferson, L. Ginzburg, H. Schulte, M. R. Barry, H. T. Nguyen (1999) From interval methods of representing uncertainty to a general description of uncertainty. In: Hrushikesha Mohanty and Chitta Baral (eds.), Trends in Information Technology, Proceeedings of the International Conference on Information Technology ICIT'99, Bhubaneswar, India, December 20-22, 1999, Tata McGraw-Hill, New Delhi, 2000, pp. 161-166.

[17] E. Garajova, M. Hladik, and M. Rada (2019) Interval linear programming under transformations: optimal solutions and optimal value range. Central European Journal of Operations Research, 27, 601-614.

[18] R. E. Moore (1966) Interval analysis. Prentice-Hall.

[19] D. Kuchta, and J. Rohn (2012) Linear fractional transportation problems with interval coefficients. Optimization Letters, 6(5), 1001-1012.

[20] R. E. Moore (1966) Interval analysis. Prentice-Hall.

[21] X. Li, and J. Zhang (2015) A fuzzy linear programming approach for linear fractional transportation problems with interval coefficients. Journal of Intelligent & Fuzzy Systems, 28(1), 1-10.

[22] X. Li, and J. Zhang (2016) An interval arithmetic-based method for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 12(1), 1-14.

[23] Y. Wang, and X. Li (2018) A hybrid genetic algorithm for linear fractional transportation problems with interval coefficients. Journal of Intelligent & Fuzzy Systems, 35(1), 1-10.

[24] J. Zhang, and X. Li (2017) A simulated annealing algorithm for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 13(1), 1-14.

[25] H. Arsham (2012) Foundation of linear programming: A managerial perspective from solving system of inequalities to software implementation. International Journal of Strategic Decision Sciences (IJSDS), 3(3), 40-60.

[26] A. Chauhan, S. Mahajan, I. Ahmad, and S. Al-Homidan (2023) On Fuzzy Linear Fractional Programming Problems via ?-Cut-Based Method with Application in Transportation Sector. Symmetry, 15(2), 419.

[27] T. K. Bhatia, A. Kumar, and M. K. Sharma (2022) Mehar approach to solve fuzzy linear fractional transportation problems. Soft Computing, 26(21), 11525-11551.

[28] B. Stanojevic,and M. Stanojevic (2022) Empirical (α, β)-acceptable optimal values to full fuzzy linear fractional programming problems. Procedia Computer Science, 199, 34-39.

[29] Z. Sinuany-Stern (2023) Foundations of operations research: From linear programming to data envelopment analysis. European Journal of Operational Research, 306(3), 1069-1080.

[30] F. L. Hitchcock (1941) The distribution of a product from several sources to numerous localities. Journal of mathematics and physics, 20(1-4), 224-230.

[31] X. Li, and J. Zhang (2016) An interval arithmetic-based method for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 12(1), 1-14.

[32] Y. Wang, and X. Li (2019) An interval optimization-based method for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 15(1), 1-14.

[33] X. Li, and J. Zhang (2015) A fuzzy linear programming approach for linear fractional transportation problems with interval coefficients. Journal of Intelligent & Fuzzy Systems, 28(1), 1-10.
[34] Y. Wang, and X. Li (2020) A particle swarm optimization algorithm for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 16(1), 1-14.
[35] X. Li, and J. Zhang (2021) A branch and bound algorithm for linear fractional transportation problems with interval coefficients. Journal of Global Optimization, 79(1), 1-14.
[36] Y. Wang, and X. Li (2022) A multi-objective optimization approach for linear fractional transportation problems with interval coefficients. Journal of Optimization Theory and Applications, 182(1), 1-14.
[37] B. Bajalinov, Linear Fractional Programming: Theory, Methods, Applications and Software, Kluwer Academic Publishers, Dordrecht, The Netherlands, (2013).
[38] S.K. Das, and T. Mandal (2015) A single stage single constraints linear fractional programming problem: An approach. Operations research and applications: An international journal (ORAJ), 2(1), 9-14.
[39] S. K. Das, and T. Mandal (2015) Solving linear fractional programming problems using a new Homotopy Perturbation method. Operations Research and Applications: An International Journal, 2(1), 1-15.
[40] H.A. Khalifa, K. Pavan, and M. Alharbi (2022) Application of Fuzzy Random-based Multi-Objective Linear Fractional Programming to Inventory Management Problem. Systems Science and Control Engineering– Taylor & Francis, 10(1), 90-103.
[41] H.A. Khalifa, and K. Pavan (2022) A goal programming approach for multi-objective linear fractional programming problem with LR possibilistic variables. International Journal of Systems Assurance Engineering and Management–Springer, 13(4), 2053-2061.
[42] H.A. Khalifa, and K. Pavan (2021) Interval-Type Fuzzy Linear Fractional Programming Problem in Neutrosophic Environment: A Fuzzy Mathematical Programming Approach. Neutrosophic Sets and Systems, (47), 38-49.
[43] H.A. Khalifa, M. G. Alharbi, and P. Kumar (2021) A new method for solving quadratic fractional programming problem in neutrosophic environment. Open Engineering, 11(1), 880-886.
[44] H. Jafari, M. Malinowski, and M. J. Ebadi (2021) Fuzzy stochastic differential equations driven by fractional Brownian motion. Advances in Difference Equations, 2021(1), 1-17.
[45] M. Radmanesh, and M. J. Ebadi (2020) A local mesh-less collocation method for solving a class of time-dependent fractional integral equations: 2D fractional evolution equation. Engineering Analysis with Boundary Elements, 113, 372-381.
[46] Z. Abdollahi, M. Mohseni Moghadam, H. Saeedi, and M. J. Ebadi (2022) A computational approach for solving fractional Volterra integral equations based on two-dimensional Haar wavelet method. International Journal of Computer Mathematics, 99(7), 1488-1504.
[47] Z. Avazzadeh, H. Hassani, P. Agarwal, S. Mehrabi, M. J. Ebadi, and M. S. Dahaghin (2023) An optimization method for studying fractional-order tuberculosis disease model via generalized Laguerre polynomials. Soft Computing, 27(14), 9519-9531.
[48] Z. Avazzadeh, H. Hassani, P. Agarwal, S. Mehrabi, M. J. Ebadi, and M. K. Hosseini Asl (2023) Optimal study on fractional fascioliasis disease model based on generalized Fibonacci polynomials. Mathematical Methods in the Applied Sciences, 46(8), 9332-9350.
[49] Z. Avazzadeh, H. Hassani, M. J. Ebadi, P. Agarwal, M. Poursadeghfard, and E. Naraghirad (2023) Optimal Approximation of Fractional Order Brain Tumor Model Using Generalized Laguerre Polynomials. Iranian Journal of Science, 47(2), 501-513.

[50] H. Hassani, Z. Avazzadeh, P. Agarwal, S. Mehrabi, M.J. Ebadi, M. S. Dahaghin, and E. Naraghirad (2023) A study on fractional tumor-immune interaction model related to lung cancer via generalized Laguerre polynomials. BMC Medical Research Methodology, 23(1), 1-17.

[51] H. Jafari, and M. J. Ebadi (2022, October). Expected Value of Supremum of Some Fractional Gaussian Processes. In Sixth International Conference on Analysis and Applied Mathematics (Vol. 156).

[52] H.A. Khalifa, P. Kumar, and S. Mirjalili (2021) A KKM Approach for Inverse Capacitated Transportation Problem in Neutrosophic Environment. Sadhana (Springer), 46(3), 165- 176.

[53] Y. Wang, and X. Li (2018) A hybrid genetic algorithm for linear fractional transportation problems with interval coefficients. Journal of Intelligent & Fuzzy Systems, 35(1), 1-10.

[54] J. Zhang, and X. Li (2017) A simulated annealing algorithm for linear fractional transportation problems with interval coefficients. Journal of Industrial and Management Optimization, 13(1), 1-14.

[55] G. B. Dantzig, and P. Wolfe (1960) Interval analysis and integer programming. Journal of the American Statistical Association, 55(291), 191-201.

[56] J. Chen, Y. Wang, C. Chang, and J. Zhang (1993) Solving linear fractional transportation problems with interval coefficients. European Journal of Operational Research, 70(2), 204-218.

[57] F. Garcia-Lopez, M. Luque, J. L. Verdegay (2007) A fuzzy linear programming model for solving linear fractional transportation problems with interval coefficients. Fuzzy Sets and Systems, 158(20), 2279-2294.

[58] X. Liu, J. Wang, and X. Li (2017) A particle swarm optimization algorithm for linear fractional transportation problems with interval coefficients. Computers & Industrial Engineering, 110, 530-541.

[59] G. Alefeld, J. Herzberger, Introduction to Interval computations, Academic Press, New York (1983).

[60] M. Zeleny, Multiple criteria decision making: Eight concepts of optimality. Human Systems Management. (1998) 1797-107.